B and D meson leptonic decay constants and quark masses from four-flavor lattice QCD

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Outline

1. HISQ ensembles with (2+1+1)-flavors of dynamical quarks
2. Decay constants of heavy-light mesons
3. Extraction of quark masses from heavy-light meson masses

References

- Decay constants: arXiv:1712.09262 (Fermilab Lattice/MILC Collaborations)
- Quark masses: arXiv:1802.04248 (Fermilab Lattice/MILC/TUMQCD Collaborations)
- Both topics: Javad Komijani, Lattice 2017 Conference (Granada).

Motivation

- Need precise values of decay constants f_B and f_{B_s} to
  - provide accurate Standard Model predictions for rare decays such as $B \to \tau\nu$, neutral B mixing, and $B_s \to \mu^+\mu^-$.
  - probe the $V - A$ structure of the W_{ub} vertex.
  - resolve/sharpen tension between inclusive and exclusive $|V_{ub}|$.
- Need precise values of quark masses
  - for precise SM predictions.
  - to test Higgs origin of quark masses.

How we achieve high precision

- Gluon gauge-field configurations generated with the highly-improved staggered quark (HISQ) formulation for sea quarks.
- HISQ formulation for all valence quarks, including b, following HPQCD [Phys. Rev. **D85**, 031503 (2012).]
- Exploit merger of effective theories to carry out extrapolations to the physical point:
  - Heavy-quark effective theory (HQET) to treat heavy-quark discretization effects.
  - Chiral perturbation theory (HMrPQASχPT) to treat the light-quark mass dependence.
  - Symanzik effective theory (SET) to treat light-quark and gluon discretization.
- New minimal renormalon subtraction (MRS) scheme improves HQET for quark mass calculation. [Phys. Rev. **D97**, 034503 (2018)], [Komijani, JHEP **08**, 062 (2017)].
- High statistics: 24 gauge-field ensembles with approximate lattice spacing ranging from 0.03 to 0.15 fm, several values of the light quark masses, including physical values.

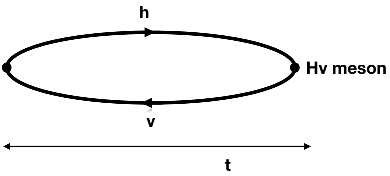
MILC ensembles with (2+1+1)-flavors of dynamical quarks

- Ensembles with physical mass for the strange quark:

$\approx a$ (fm)	m_l/m_s	size	L (fm)	$M_\pi L$	M_π (MeV)
0.15	1/5	$16^3 \times 48$	2.38	3.8	314
0.15	1/10	$24^3 \times 48$	3.67	4.0	214
0.15	1/27	$32^3 \times 48$	4.83	3.2	130
0.12	1/5	$24^3 \times 64$	3.00	4.5	299
0.12	1/10	$24^3 \times 64$	2.89	3.2	221
0.12	1/10	$32^3 \times 64$	3.93	4.3	216
0.12	1/10	$40^3 \times 64$	4.95	5.4	214
0.12	1/27	$48^3 \times 64$	5.82	3.9	133
0.09	1/5	$32^3 \times 96$	2.95	4.5	301
0.09	1/10	$48^3 \times 96$	4.33	4.7	215
0.09	1/27	$64^3 \times 96$	5.62	3.7	130
0.06	1/5	$48^3 \times 144$	2.94	4.5	304
0.06	1/10	$64^3 \times 144$	3.79	4.3	224
0.06	1/27	$96^3 \times 192$	5.44	3.7	135
0.042	1/5	$64^3 \times 192$	2.91	4.34	294
0.042	1/27	$144^3 \times 288$	6.12	4.17	134
0.03	1/5	$96^3 \times 288$	3.25	4.84	294

- The fermion action is "highly improved staggered quark" (HISQ) action
- Physical-mass ensembles at most lattice spacings

Heavy-Light Pseudoscalar Mesons

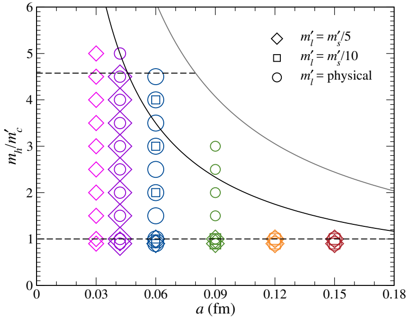


- Decay constant (Notation: "H_{v}" instead of "B^+" or "B_s"when masses and light quarks vary)
- Pseudoscalar density-density correlator

$$C_{\mathrm{pt-pt}}(t) \to A_{\mathrm{pt-pt}} \exp(-M_{H_{\mathsf{v}}} t) \tag{1}$$

$$F_{H_{\mathsf{v}}} = (m_h + m_{\mathsf{v}}) \sqrt{\frac{3V A_{pt-pt}}{2M_{H_{\mathsf{v}}}^3}} \tag{2}$$

- light valence: $\frac{m_u+m_d}{2} \equiv m_l \lesssim m_{\mathsf{v}} \lesssim m_s$
- heavy valence: $m_c \lesssim m_h \lesssim m_b$



- Plot symbols indicate the range of heavy valence quark masses used for each lattice spacing and light sea quark mass ratio.
- We use only $am_h < 0.9$ to avoid large discretization errors

Scale setting and calculating tuned quark masses

- Intermediate scale setting for chiral analysis is done using f_{p4s}
  It is the decay constant of a fictitious pseudoscalar meson with both valence masses equal to $m_{p4s} \equiv 0.4 m_s$.
- The physical value of f_{p4s} is set from f_π.
- This method yields a precise determination of both the lattice spacing a and the quark mass am_{p4s} (and in turn $m_s = 2.5 m_{p4s}$)
- The values of f_{p4s} and quark mass ratio m_s/m_l are determined by analyzing light-light data from the same ensembles
  $\Rightarrow$ Various systematic errors (such as FV, EM, continuum extrapolation *etc.*) in estimate of f_{p4s} and tuned quark masses must be incorporated to our estimate of uncertainties

Constructing fit function for decay constants

- We use a cascade of EFTs to construct our fit functions
- We start from the following schematic form for decay constants of H_{v} mesons

$$f_{H_{\mathrm{v}}}\sqrt{M_{H_{\mathrm{v}}}} \equiv \Phi_{H_{\mathrm{v}}} = C\ (1+\mathrm{SET})\ (1+\mathrm{HQET})\ (1+\mathrm{HMrAS}\chi\mathrm{PT}) \left(\frac{m_c'}{m_c}\right)^{3/27} \tilde{\Phi}_0$$

- These terms correspond to different effective field theories

Symanzik Effective Theory (SET)

$$c_1\alpha_s\,(a\Lambda_{\mathrm{QCD}})^2 + \cdots + c_3\alpha_s\,(am_h)^2 + \cdots$$

Wilson coefficient C

$$\left[\alpha_s(M_{H_s})\right]^{-6/25}\left(1+\mathcal{O}(\alpha_s)\right)$$

HQET (and integrating out sea-charm)

$$k_1\frac{\Lambda_{\mathrm{HQET}}}{M_{H_s}} + \cdots + k_1'\frac{m_c}{m_c'}$$

Integrating out charm quark

$$\frac{\Lambda^{(3)}_{\mathrm{QCD}}(m_c')}{\Lambda^{(3)}_{\mathrm{QCD}}(m_c)} \approx \left(\frac{m_c'}{m_c}\right)^{2/27}$$

HMrPQASχPT at NLO

chiral non-analytic terms
$+L_{\mathrm{v}}m_{\mathrm{v}} + L_{\mathrm{s}}(2m_l+m_s) + L_a a^2$

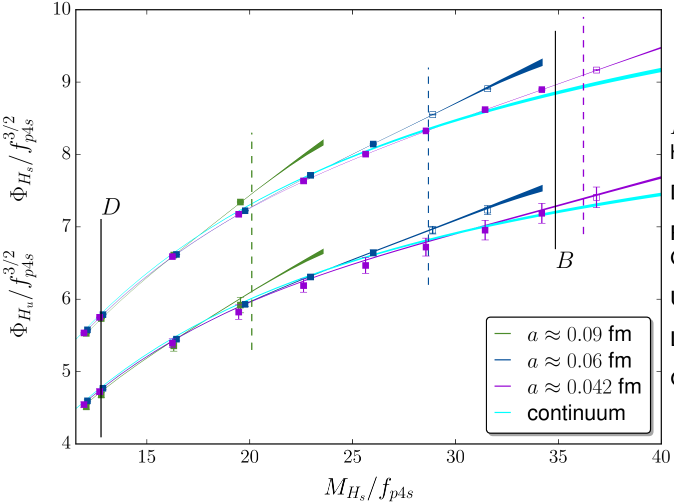
Chiral terms contain effects of

- Taste splittings in "pion" masses & new logs.
- Hyperfine and flavor splittings
- Finite lattice volume

Constructing fit function for decay constants

- As indicated in the previous slide, to take into account SET, higher order analytic χPTterms, and higher order HQET effects, we include analytic terms.
- They are typically polynomials in dimensionless, "natural" expansion parameters:
  - Light-quark and gluon discretization (SET): $(a\Lambda_{\mathrm{QCD}})^2$ with $\Lambda_{\mathrm{QCD}} = 600$ MeV.
  - Heavy-quark discretization effects (SET-HISQ): $(2am_h/\pi)^2$
  - Light valence and sea quark mass effects (χPT): $B_0/(4\pi^2 f_\pi^2)\, m_{\mathrm{v}}$
  - HQET $\Lambda_{\mathrm{HQET}}/M_{H_s}$.
- The coefficients of the polynomials are fit parameters. Expected to be $\mathcal{O}(1)$.
- 60 fit parameters – 492 data points.

A snapshot of the fit and data



M_{Hs} as proxy of heavy-quark mass m_h

Dashed lines: $am_h = 0.9$

Filled Symbols: included
Open Symbols: excluded

Upper curves: $m_v = m_s$

Lower curves: $m_v = m_u$

Cyan bands: extrap. to cont.

χ^2/d.o.f = 466/432, p value=0.12

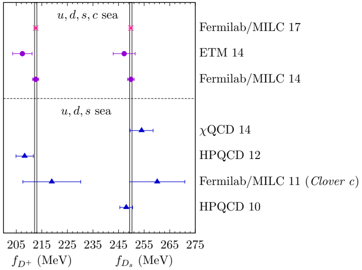
Results for decay constants

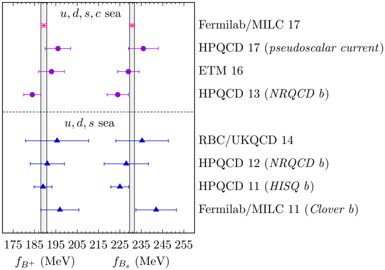
$$
\begin{aligned}
f_{D^0} &= 211.5 \pm 0.3_{\rm stat} \pm 0.3_{\rm sys} \pm 0.2_{f_\pi,{\rm PDG}}\ {\rm MeV} \\
f_{D^+} &= 212.6 \pm 0.3_{\rm stat} \pm 0.3_{\rm sys} \pm 0.2_{f_\pi,{\rm PDG}}\ {\rm MeV} \\
f_{D_s} &= 249.8 \pm 0.3_{\rm stat} \pm 0.3_{\rm sys} \pm 0.2_{f_\pi,{\rm PDG}}\ {\rm MeV} \\
f_{B^+} &= 189.4 \pm 0.8_{\rm stat} \pm 1.1_{\rm sys} \pm 0.3_{f_\pi,{\rm PDG}}\ {\rm MeV} \\
f_{B^0} &= 190.5 \pm 0.8_{\rm stat} \pm 1.0_{\rm sys} \pm 0.3_{f_\pi,{\rm PDG}}\ {\rm MeV} \\
f_{B_s} &= 230.7 \pm 0.8_{\rm stat} \pm 0.8_{\rm sys} \pm 0.2_{f_\pi,{\rm PDG}}\ {\rm MeV}
\end{aligned}
$$

The systematic error includes

- Continuum extrapolation
- Finite volume
- EM contribution to meson masses that are used to fix the quark masses (Decay constants are pure-QCD quantities; EM contributions to the relation between decay constants and physical decay rates are not included here by definition but would be relevant for phenomenology)
- Uncertainty in adjustment for non-equilibration of topological charge

Comparison with previous 3 and 4 flavor calculations





Our results are shown in red. The gray bands indicate the total error.

Extraction of quark masses from heavy-light meson masses

- New method based on HQET to extract masses of quarks from masses of heavy-light mesons

$$M_H = m_h + \bar{\Lambda} + \frac{\mu_\pi^2 - \mu_G^2(m_h)}{2m_h} + \mathcal{O}(1/m_h^2)$$

  - $\bar{\Lambda}$: energy of light quark and gluons inside the system
  - $\mu_\pi^2/2m_h$: kinetic energy of the heavy quark inside the system
  - $\mu_G^2(m_h)/2m_h$: hyperfine energy due to heavy quark's spin (can be estimated from B^*-B splitting $\Rightarrow \mu_G^2(m_b) \approx 0.35\,\mathrm{GeV}^2$)
  - m_h is the pole mass of the heavy quark

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  - m_h is the pole mass of the heavy quark (Conventional pole mass is ambiguous because of the renormalon problem)
  - We use the new minimal renormalon-subtracted (MRS) mass [Brambilla *et al.* Phys. Rev. **D97**, 034503 (2018); Komijani, JHEP **08**, 062 (2017)].
  - Removes the leading infrared renormalon from the pole mass
  - It is a gauge- and scale-independent scheme
  - Admits a well-behaved perturbative expansion in α_s

Mapping bare quark masses to the $\overline{\text{MS}}$ and MRS masses

1) Continuum relation between the MRS and $\overline{\text{MS}}$ mass:

$$m_{\text{MRS}} = \bar{m}\left(1 + \sum_{n=0}^{\infty}[r_n - R_n]\alpha_s^{n+1}(\bar{m}) + J_{\text{MRS}}(\bar{m})\right) \quad (3)$$

- $\bar{m} \equiv m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}})$ and $J_{\text{MRS}}(\bar{m})$ is known [arXiv:1712.04983].
- Small coefficients $[r_n - R_n]$ are the difference between the $\overline{\text{MS}}$ and MRS expansion.

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2) Introduce a "reference mass", $m_r = m_{p4s,\overline{\text{MS}}}(\mu)$. Choose $\mu = 2\text{GeV}$. Identity:

$$m_{h,\text{MRS}} \equiv m_{r,\overline{\text{MS}}}(\mu) \frac{m_{h,\text{MRS}}}{\bar{m}_h} \frac{\bar{m}_h}{m_{r,\overline{\text{MS}}}(\mu)} \frac{am_h}{am_r} \tag{4}$$

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3) First factor, a fit parameter. Second, use Eq (3) above. Third, $\overline{\text{MS}}$ mass running;

$$\frac{\bar{m}_h}{m_{r,\overline{\text{MS}}}(\mu)} = \frac{C(\alpha_{\overline{\text{MS}}}(\bar{m}_h))}{C(\alpha_{\overline{\text{MS}}}(\mu))} \tag{5}$$

Last factor from simulation parameters.

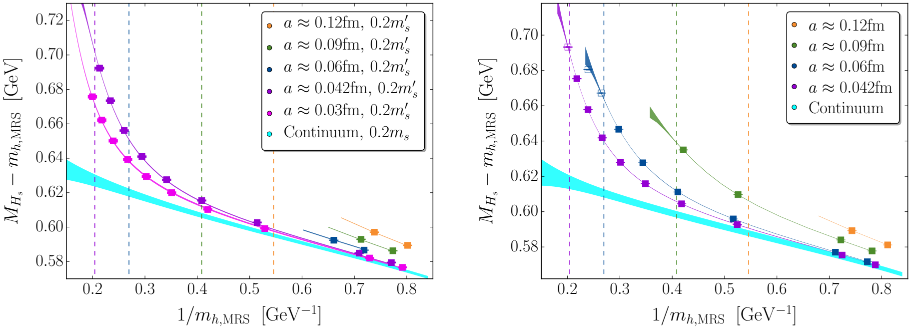
Including discretization effects, light quark mass dependence

- Include HMrPQASχPT terms

$$M_H = m_{h,\mathrm{MRS}} + \bar{\Lambda}_{\mathrm{MRS}} + \frac{\mu_\pi^2 - \mu_G^2(m_h)}{2m_{h,\mathrm{MRS}}} + \mathrm{HMrAS}\chi\mathrm{PT} + \text{higher order HQET} \quad (6)$$

- As with the decay constants, all terms get corrections for light-quark- and gluon-discretization effects, based on polynomials in the natural expansion parameters. Polynomial coefficients are fit parameters.
- 67 parameters (6 with external priors) and 384 data points
- Adjust experimentally measured masses for EM effects before matching to the continuum extrapolation of the fit result.
- Light-quark masses and decay constants use light-quark rPQSχPT.

A snapshot of the fit and data



Dashed lines: $am_h \approx 0.9$; open symbols: data points omitted from fit
Vertical axis: heavy-strange meson masses minus the (fitted) MRS heavy quark mass
Horizontal axis: inverse MRS heavy-quark mass

χ^2/d.o.f $= 312/307$, p value=0.3

- After extrapolating to continuum, measured M_{D_s} and M_{B_s} masses with EM effects removed are used to determine the charm- and bottom-quark masses.

Results for heavy quark masses

Results for heavy quark masses and their ratios in a theory with 4 active flavors

$$m_{s,\overline{\mathsf{MS}}}(2\ \mathrm{GeV}) = 92.52(40)_{\mathsf{stat}}(18)_{\mathsf{syst}}(52)_{\alpha_s}(12)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV} \qquad (7)$$

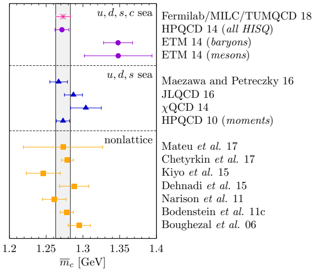
$$\begin{aligned} \overline{m}_c &= 1273(1)_{\mathrm{stat}}(10)_{\mathrm{sys}}(1)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV} \\ \overline{m}_b &= 4203(12)_{\mathrm{stat}}(8)_{\mathrm{sys}}(1)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV} \end{aligned}$$

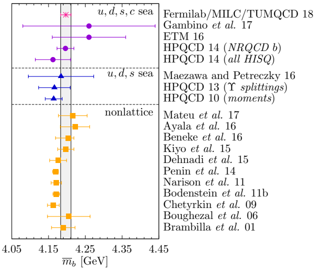
$$\begin{aligned} m_c/m_s &= 11.784(11)_{\mathsf{stat}}(17)_{\mathsf{syst}}(00)_{\alpha_s}(08)_{f_{\pi,\mathrm{PDG}}}, && (8) \\ m_b/m_s &= 53.93(7)_{\mathsf{stat}}(8)_{\mathsf{syst}}(1)_{\alpha_s}(5)_{f_{\pi,\mathrm{PDG}}}, && (9) \\ m_b/m_c &= 4.577(5)_{\mathsf{stat}}(7)_{\mathsf{syst}}(0)_{\alpha_s}(1)_{f_{\pi,\mathrm{PDG}}} && (10) \end{aligned}$$

where $\overline{m}_h = m_{h,\overline{\mathsf{MS}}}(m_{h,\overline{\mathsf{MS}}})$. The systematic error includes

- errors in the determination of scale setting quantities, quark mass tuning, continuum extrapolation, finite volume, and estimating EM effects;
- uncertainty in the strong coupling constant $\alpha_s^{\overline{MS}}(5\,\mathrm{GeV}; n_f = 4) = 0.2128(25)$ [HPQCD, arXiv:1408.4169]
- the value of f_π

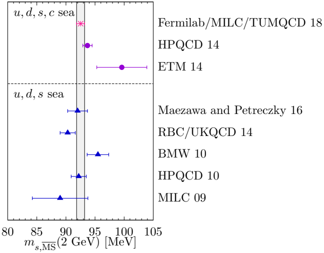
Comparison with other results

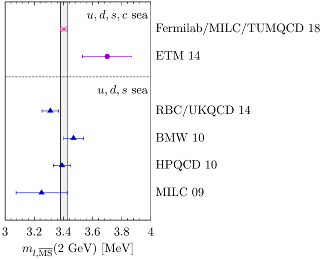




Our results are shown in red. The gray bands indicate the total error.

Results for light quark masses





Results for light quark masses in a theory with 4 active flavors.

$$m_{u,\overline{\mathsf{MS}}}(2\ \mathrm{GeV}) = 2.118(17)_{\mathsf{stat}}(32)_{\mathsf{syst}}(12)_{\alpha_s}(03)_{f_\pi,\mathrm{PDG}}\ \mathrm{MeV}, \tag{11}$$

$$m_{d,\overline{\mathsf{MS}}}(2\ \mathrm{GeV}) = 4.690(30)_{\mathsf{stat}}(36)_{\mathsf{syst}}(26)_{\alpha_s}(06)_{f_\pi,\mathrm{PDG}}\ \mathrm{MeV}, \tag{12}$$

Results for HQET LECs

Results for HQET Low-energy constants in the MRS scheme

$$\overline{\Lambda}_{\mathsf{MRS}} = 552(25)_{\rm stat}(6)_{\rm sys}(16)_{\alpha_s}(2)_{f_\pi \rm PDG}\,\rm MeV$$
$$\mu_\pi^2 = 0.06(16)_{\mathsf{stat}}(14)_{\mathsf{syst}}(06)_{\alpha_s}(00)_{f_\pi \rm PDG}\ \rm GeV^2 \quad (13)$$
$$\mu_G^2(m_b) = 0.38(01)_{\mathsf{stat}}(01)_{\mathsf{syst}}(00)_{\alpha_s}(00)_{f_\pi \rm PDG}\ \rm GeV^2$$

(14)

Note that the prior value of $\mu_G^2(m_b)$ is set to $0.35 \pm 0.07\,\rm GeV^2$ [P. Gambino and C. Schwanda, arXiv:1307.4551]

Conclusion

- Decay constants
  - We used a combination of EFTs in a correlated, multidimensional fit to lattice data at multiple lattice spacings
    $\Rightarrow$ reduces statistical errors
    $\Rightarrow$ controls systematic errors of extrapolations
  - We presented results for decay constants f_{D^+}, f_{D_s}, f_{B^+} and f_{B_s}
- Quark masses
  - We developed a method based on HQET and the MRS scheme to extract quark masses from heavy-light meson masses
  - We used a combination of EFTs to fit heavy-light meson masses .
  - We presented results for all quark masses and their ratios and for HQET LECs.

back-up slides

Minimal renormalon subtracted (MRS) scheme

- The RS mass is defined by subtracting the leading renormalon of the pole mass
- In the RS mass a finite part, in addition to the leading renormalon, is subtracted from the pole mass
- The finite part depends on the subtraction scale ν_f (factorization scale)
- In the minimal renormalon subtracted (MRS) scheme we subtract only the renormalon part of pole mass (*i.e.*, the ambiguous part of the pole mass in the Borel plane)
- Define

$$\mathcal{J}^{\mathrm{MRS}}(\mu) \equiv \frac{N_m}{2\beta_0}\, \mu\, e^{-1/[2\beta_0 \alpha(\mu)]} \sum_{n=0}^{\infty} \frac{1}{n!(n-b)} \left[\frac{1}{2\beta_0\alpha(\mu)}\right]^n$$

where $b = \frac{\beta_1}{2\beta_0^2}$ and α is defined in a scheme with $\beta(\alpha) = -\beta_0\alpha^2/(1-\frac{\beta_1}{\beta_0}\alpha)$

- To convert the quark mass and $\overline{\Lambda}$ form the MRS scheme to the RS scheme :

$$\begin{aligned} m^{\mathrm{RS}}(\nu_f) &= m^{\mathrm{MRS}} - \mathcal{J}^{\mathrm{MRS}}(\nu_f) \\ \overline{\Lambda}^{\mathrm{RS}}(\nu_f) &= \overline{\Lambda}^{\mathrm{MRS}} + \mathcal{J}^{\mathrm{MRS}}(\nu_f) \end{aligned}$$

for instance we have

$$\overline{\Lambda}_{\mathsf{MRS}} = 552(25)_{\mathsf{stat}}(6)_{\mathsf{syst}}(16)_{\alpha_s}(2)_{f_\pi,\mathsf{PDG}}\ \mathsf{MeV} \quad (15)$$

$$\overline{\Lambda}_{\mathsf{RS}} = 636(25)_{\mathsf{stat}}(6)_{\mathsf{syst}}(24)_{\alpha_s}(2)_{f_\pi,\mathsf{PDG}}\ \mathsf{MeV} \quad (16)$$

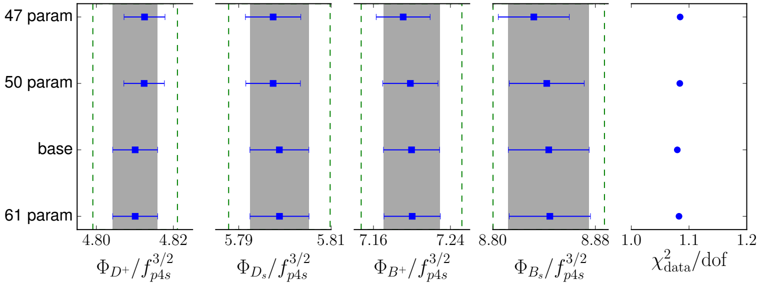
How well the renormalon subtraction works

n	r_n	R_n	$r_n - R_n$
0	0.4244	0.5350	-0.1106
1	1.0351	1.0691	-0.0340
2	3.6932	3.5966	0.0966
3	17.4358	17.4195	0.0163

The loop count is $n + 1$.

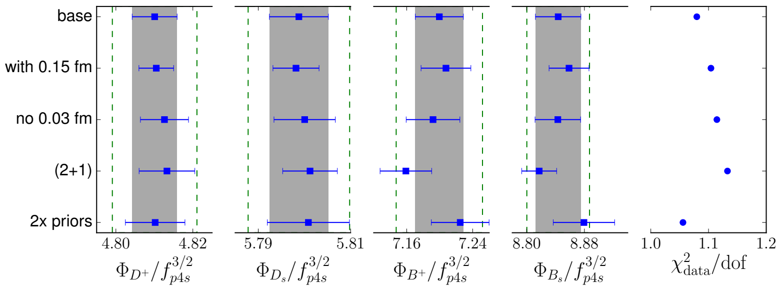
[Brambilla *et al.* Phys. Rev. **D97**, 034503 (2018); Komijani, JHEP **08**, 062 (2017)]

Stability plot for decay constants: parameter count



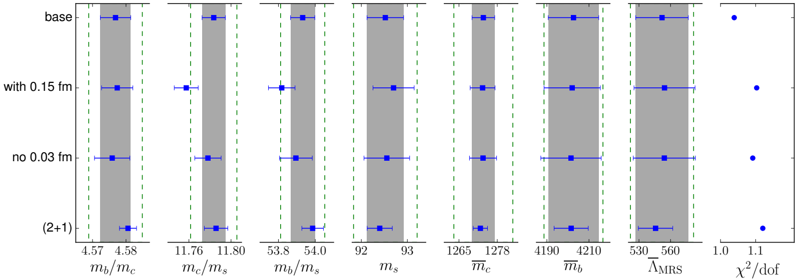
Stability plot showing the sensitivity to the number of discretization parameters. (See the text for description.) The error bars show only the statistical errors, the gray error bands correspond to the statistical error of the base fit and the green dashed lines correspond to total errors.

Stability plot for decay constants: data choices



Stability plot showing the sensitivity to different choices of lattice data. The error bars show only the statistical errors, the gray error bands correspond to the statistical error of the base fit and the green dashed lines correspond to total errors.

Stability plot for quark masses



Stability plot showing the sensitivity under variations in the data set and the form of the fit function. Here $m_s = m_{s,\overline{\mathrm{MS}}}(2\ \mathrm{GeV})$, $\overline{m}_c = m_{c,\overline{\mathrm{MS}}}(m_{c,\overline{\mathrm{MS}}})$, and $\overline{m}_b = m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}})$. The error bars show only the statistical errors, the gray error bands correspond to the statistical error of the base fit, and the dashed green lines correspond to total errors.

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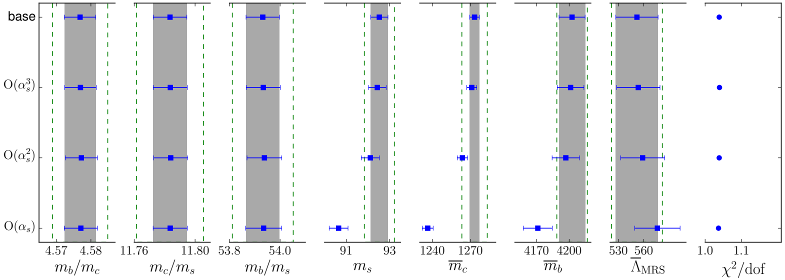

Stability plot showing the sensitivity to truncation error in perturbative-QCD relations that are used in our analysis. In the base fit, the perturbative series are accurate through order α_s^4. In the fits labeled by $\mathcal{O}(\alpha_s^n)$, we keep n subleading orders. Here $m_s = m_{s,\overline{\rm MS}}(2\ {\rm GeV})$, $\overline{m}_c = m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}})$, and $\overline{m}_b = m_{b,\overline{\rm MS}}(m_{b,\overline{\rm MS}})$. The error bars show only the statistical errors, the gray error bands correspond to the statistical error of the base fit, and the dashed green lines correspond to total errors.

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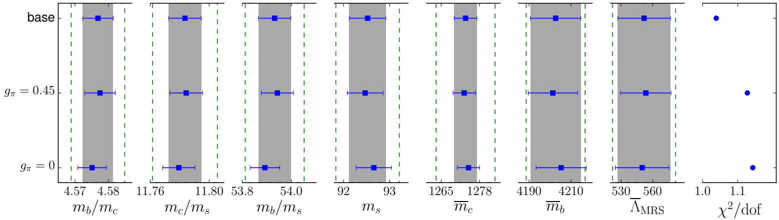

Stability plot showing the sensitivity to different choices for g_π. The error bars show only the statistical errors, the gray error bands correspond to the statistical error of the base fit and the dashed green lines correspond to total errors.