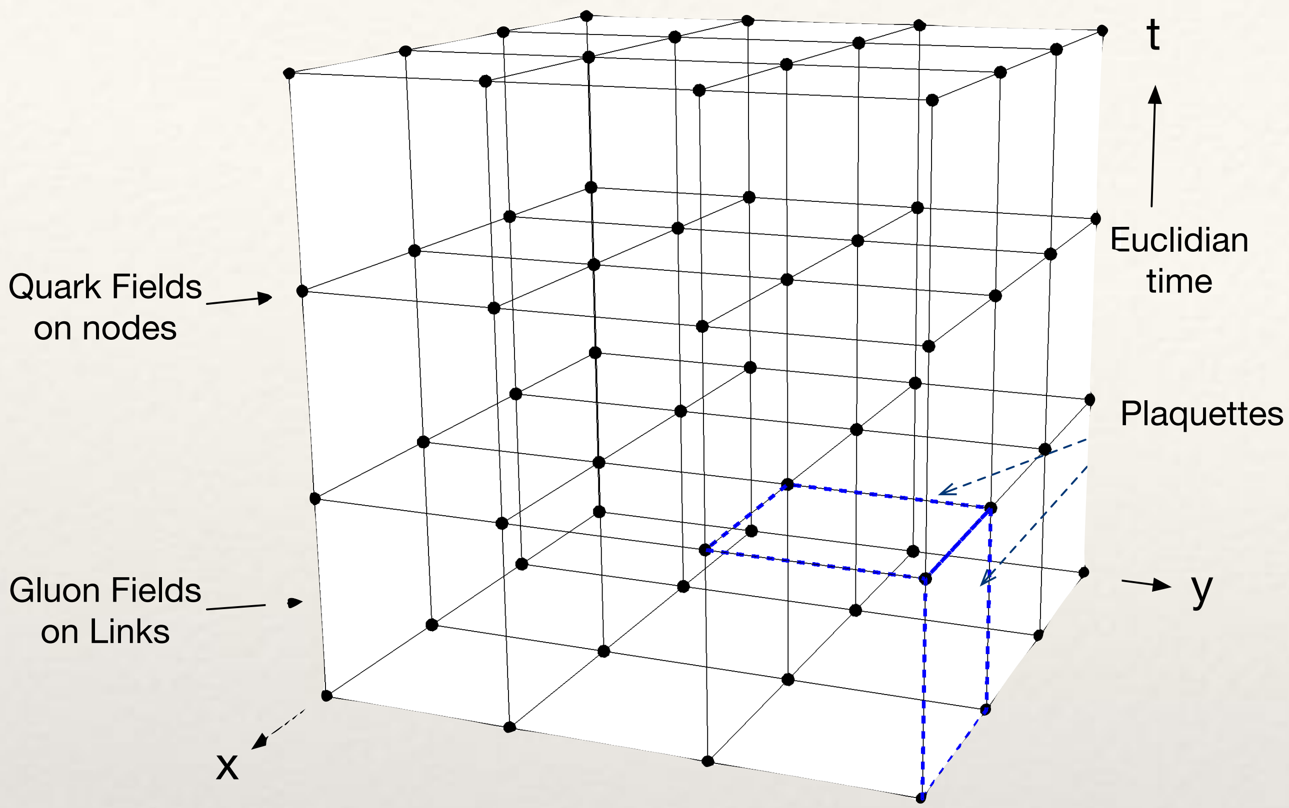

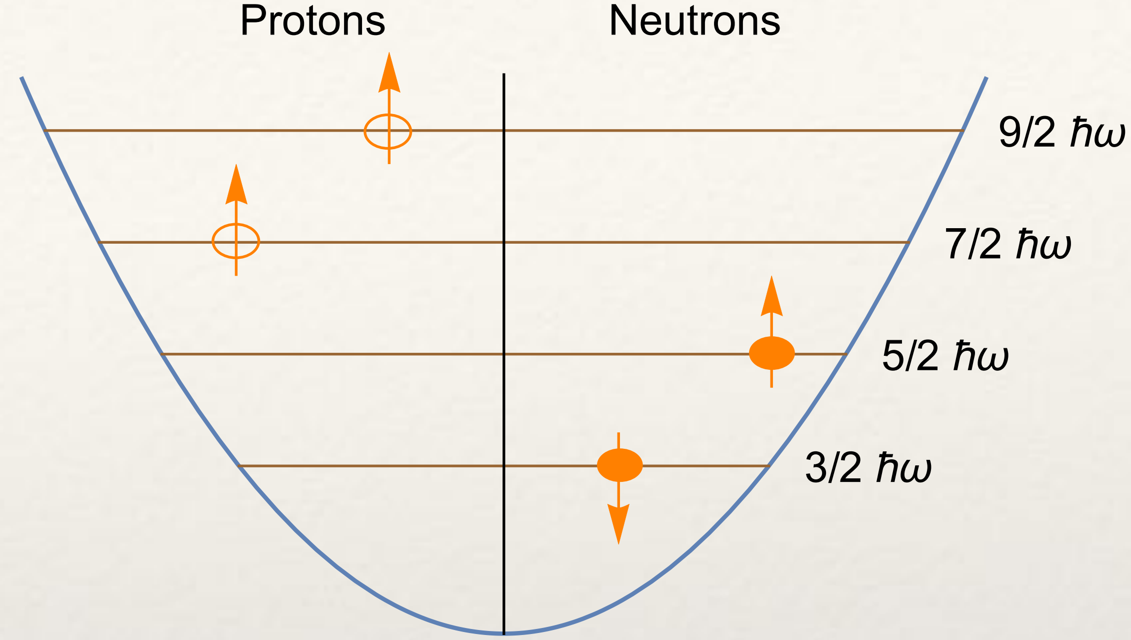

HOBET: The SM as an Effective Theory and its Direct Matching to LQCD

Kenneth S. McElvain

Berkeley
UNIVERSITY OF CALIFORNIA

BERKELEY LAB

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Nuclear Structure Calculations

- Configuration interaction calculations use an explicitly anti-symmetric basis of Slater determinants over a single particle basis.
- While the basis size grows very fast with the size of the single particle basis and A, the number of particles, fantastically efficient matrix techniques can be used to find the low lying spectrum.
- The required calculation cutoff on the basis ignores scattering through excluded states. This requires an *effective* interaction constructed in the HO basis that takes such scattering into account.

ET and The Harmonic Oscillator Basis

- We define a projection operator P for the states we will use in calculations and it's complement Q=1-P for the rest.
- An effective theory relies on a separation of scales or a weak coupling between P and Q.
- In a typical EFT using a momentum basis the kinetic energy T is diagonal and does not couple P & Q.
- In contrast, in the HO basis T is a hopping operator, strongly connecting the highest state in P to the lowest Q state.
  - Bad news for an ET expansion.
  - Maybe H_{eff} can be reorganized, isolating T …

The Bloch-Horowitz Equation

P *is projection operator to subspace to work in*, $\quad Q = 1 - P$

$$H_{eff}(E_i)|\psi_i\rangle = P\left(H + H\frac{1}{E_i - QH}QH \right)P|\psi_i\rangle = E_i P|\psi_i\rangle$$

- Eigenstates of $H_{eff}(E)$ are projections with the same eigenvalues.
  - All eigenstates that overlap P are included!
- It is continuous in energy, including across E=0. An effective theory based on the BH equation can be fit in the continuum and used to find bound states.
- Eigenstates are not orthogonal.
- Explicitly energy dependent: Must solve self consistently.
- Operators are formally renormalized as:

$$\hat{O}^{eff}_{ji} = \frac{E_j}{E_j - HQ}\hat{O}\frac{E_i}{E_i - QH}$$

The Effective Theory Expansion

The Haxton-Luu form of the Bloch-Horowitz Equation

$$H_{eff}(E) = P\frac{E}{E-TQ}\left[T + T\frac{Q}{E}T + V + V\frac{1}{E-QH}QV\right]\frac{E}{E-QT}P$$

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$$H_{eff}(E) = P\frac{E}{E-TQ}\left[T + T\frac{Q}{E}T + \underbrace{V + V\frac{1}{E-QH}QV}_{\overbrace{V_{IR}+V_{\delta}}}\right]\frac{E}{E-QT}P$$

$V_{IR} + V_{\delta}$ *ET Substitution*

The Effective Theory Expansion

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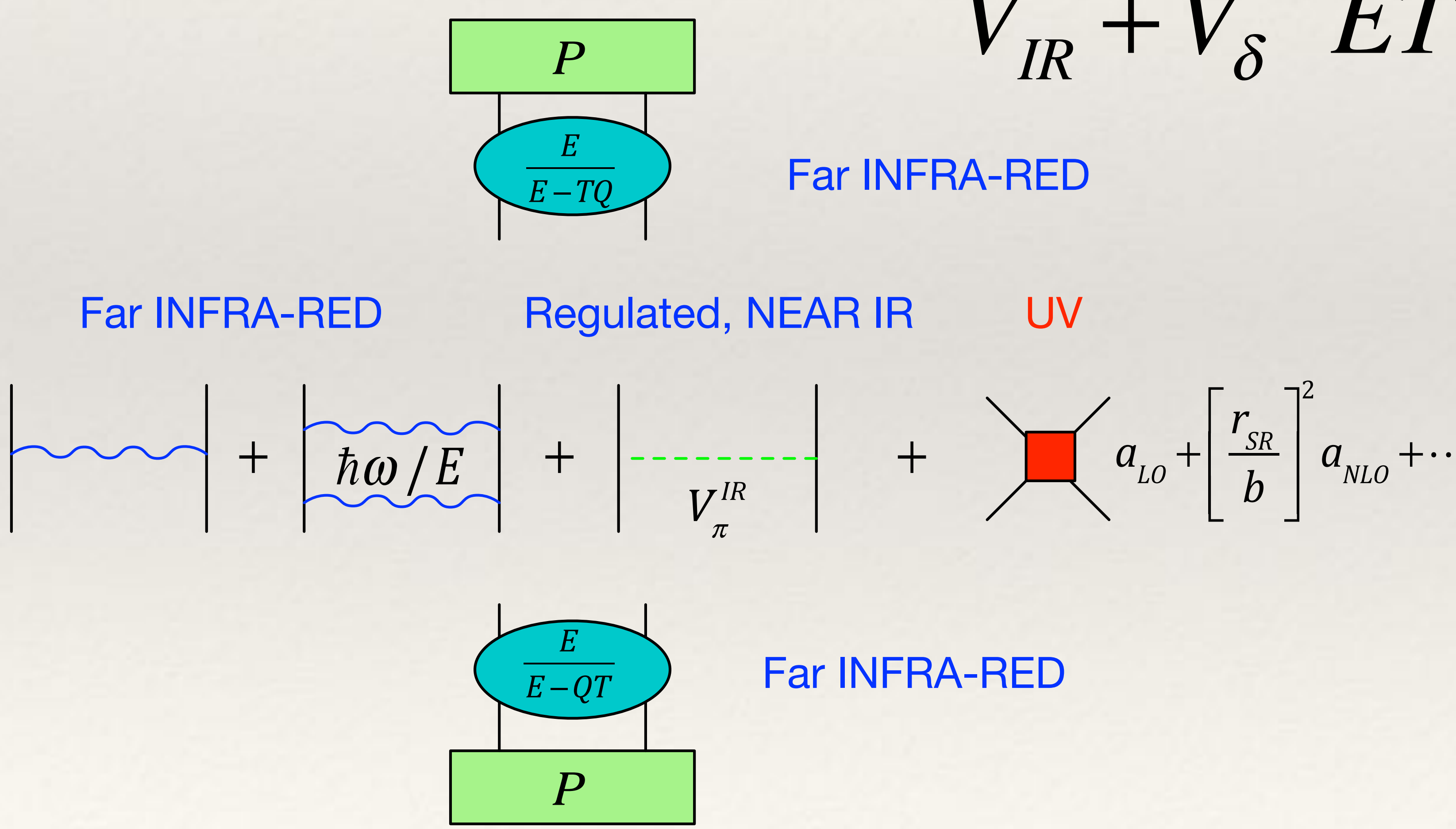
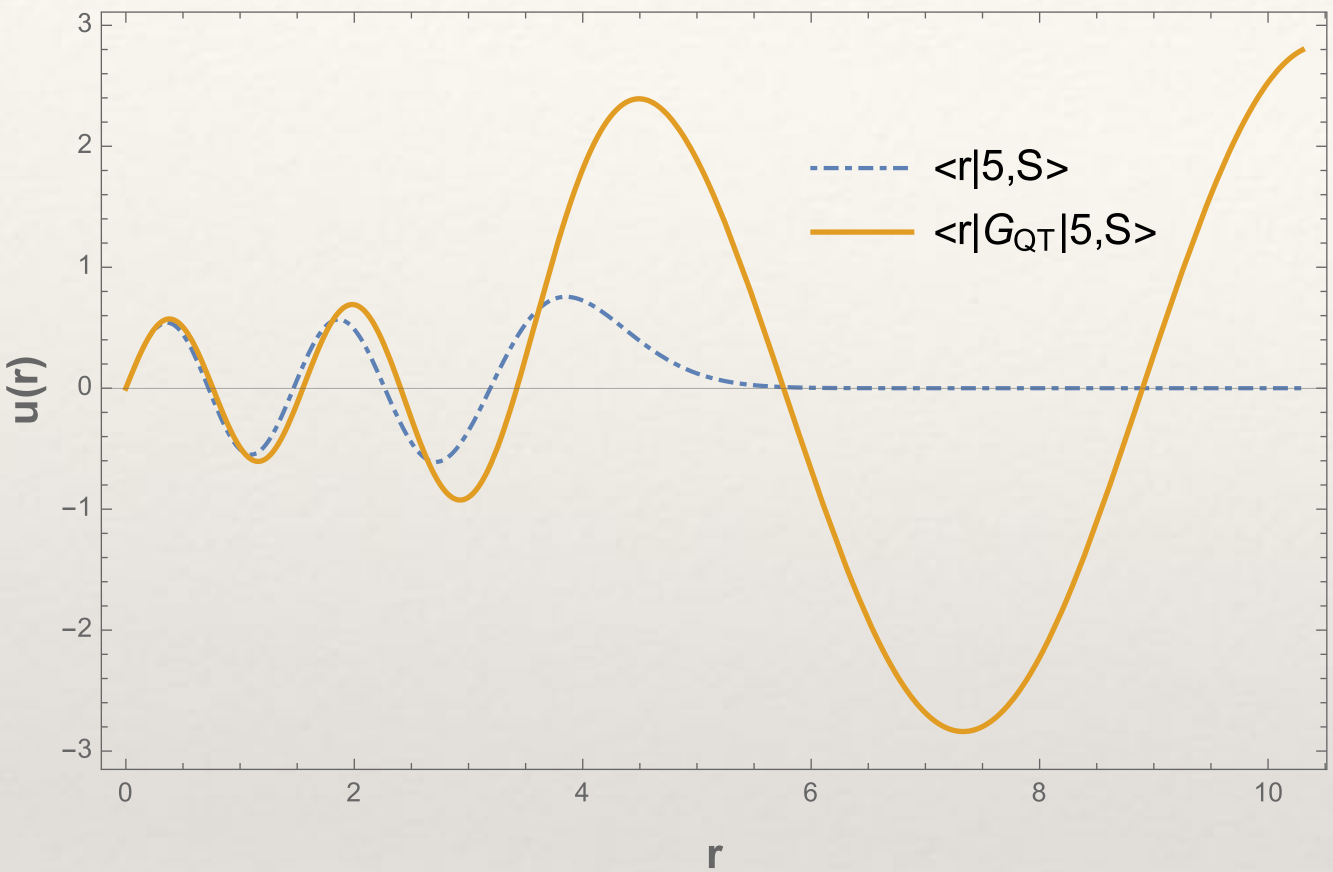
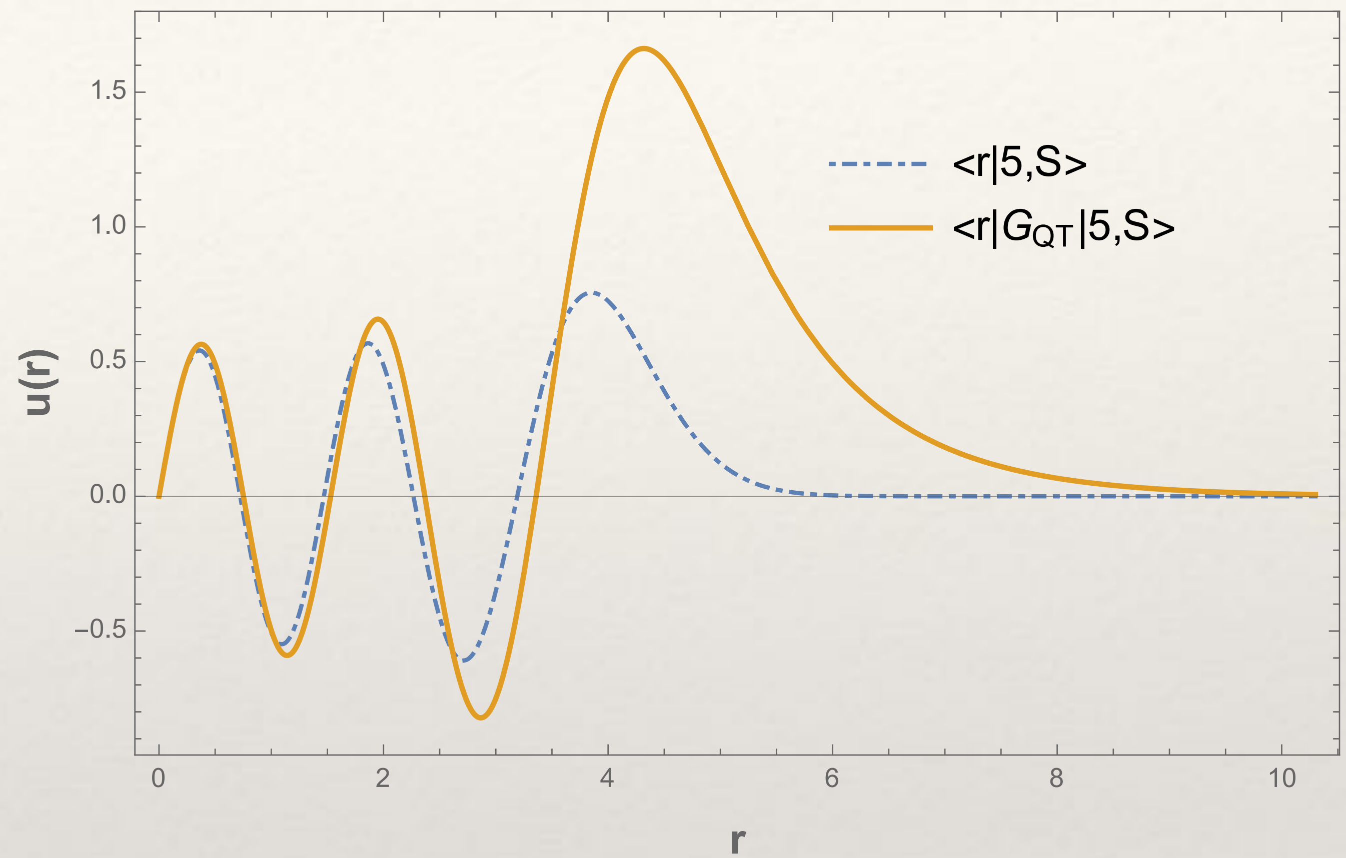


E/(E-QT) Transform of Edge States



- Acting on edge state with
  $E/\hbar\omega = 1/2$.

  Recovers scattering wave function with phase shift.



- Acting on edge state with
  $E/\hbar\omega$=-1/2.

  Recovers bound state exponential decay from gaussian falloff of HO state.

- **E/(E-QT) with boundary condition recovers IR behavior.**

Sum T to All Orders

- T contributions can be summed to all orders.

$$\left\langle j \left| \frac{E}{E-TQ}\left[T+T\frac{Q}{E}T\right]\frac{E}{E-QT} \right| i \right\rangle = E\left(\delta_{ji} - b_{ji}\right)$$

$$b_{ij} = \left\{ P\frac{E}{E-T}P \right\}_{ij}^{-1}$$

- A surprisingly simple result.
- A non-perturbative sum of kinetic energy scattering is key to a convergent ET expansion of the remaining parts.

The V_δ Expansion

- V_δ is described in terms of HO lowering operators.

$$\hat{c}\ lowers\ L,\ \hat{a}\ lowers\ nodal\ n,\quad \left[\hat{c},\hat{a}\right]=0$$

$$V_\delta^S = a_{LO}^S \delta(r) + a_{NLO}^S\left(\hat{a}^\dagger \delta(r) + \delta(r)\hat{a}\right) + \ldots$$

$$V_\delta^{SD} = a_{NLO}^{SD}\left(\hat{c}^{\dagger 2}\delta(r) + \delta(r)\hat{c}^2\right) + a_{NNLO}^{22,SD}\left(\hat{c}^{\dagger 2}\delta(r)a + \hat{a}^\dagger\delta(r)\hat{c}^2\right)$$

$$+ a_{NNLO}^{40,SD}\left(\hat{c}^{\dagger 2}\hat{a}^\dagger\delta(r) + \delta(r)\hat{a}\hat{c}^2\right) + \ldots$$

$$\ldots$$

- This is slightly simplified by absorbing a constant related to coupling spins to angular momentum into the LECs.

Matrix Structure: $^1S_0$, Λ=8

$$\left\langle \tilde{j} \middle| V_{\delta, a_{LO}} \middle| \tilde{i} \right\rangle^S = a_{LO}^S \pi^{-3/2} \begin{bmatrix} 1 & \sqrt{3/2} & \sqrt{15/8} & \sqrt{35/16} & 0.947 \\ \sqrt{3/2} & 3/2 & \sqrt{45/16} & \sqrt{105/64} & 1.160 \\ \sqrt{15/8} & \sqrt{45/16} & 15/8 & \sqrt{105/128} & 1.297 \\ \sqrt{35/16} & \sqrt{105/64} & \sqrt{105/128} & 35/16 & 1.401 \\ 0.947 & 1.160 & 1.297 & 1.401 & 0.898 \end{bmatrix}$$

- Edge state matrix elements in red vary with E due to Green's function action on edge states.
- Each such matrix corresponds to a pair (E_i, Bdy_i).

Fitting LECs

- Principle: The BH equation is energy self consistent

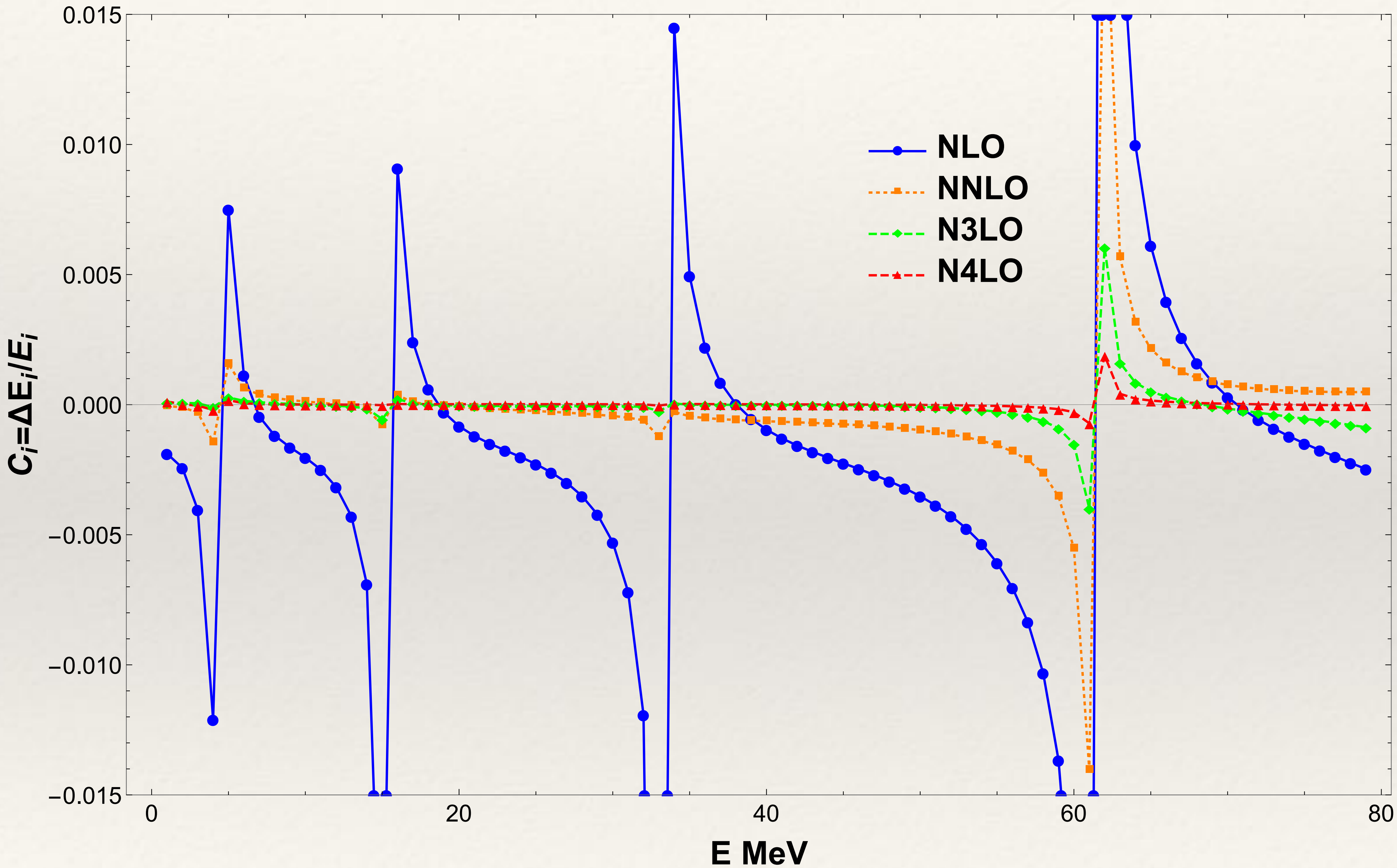
$$H_{eff}^{full}\ P\left|\psi_i\right\rangle = E_i P\left|\psi_i\right\rangle$$

- At fixed order we have a nearby eigenstate.

$$H_{eff}\left(LECs\right) P\left|\psi_i'\right\rangle = \varepsilon_i P\left|\psi_i'\right\rangle$$

- The mismatch must be due to LEC values.
- Repair by minimizing $\sum_{i \in samples} W\left(i\right)\left(\varepsilon_i - E_i\right)^2 / \sigma_i^2$
- The variance can be replaced by a full covariance matrix.

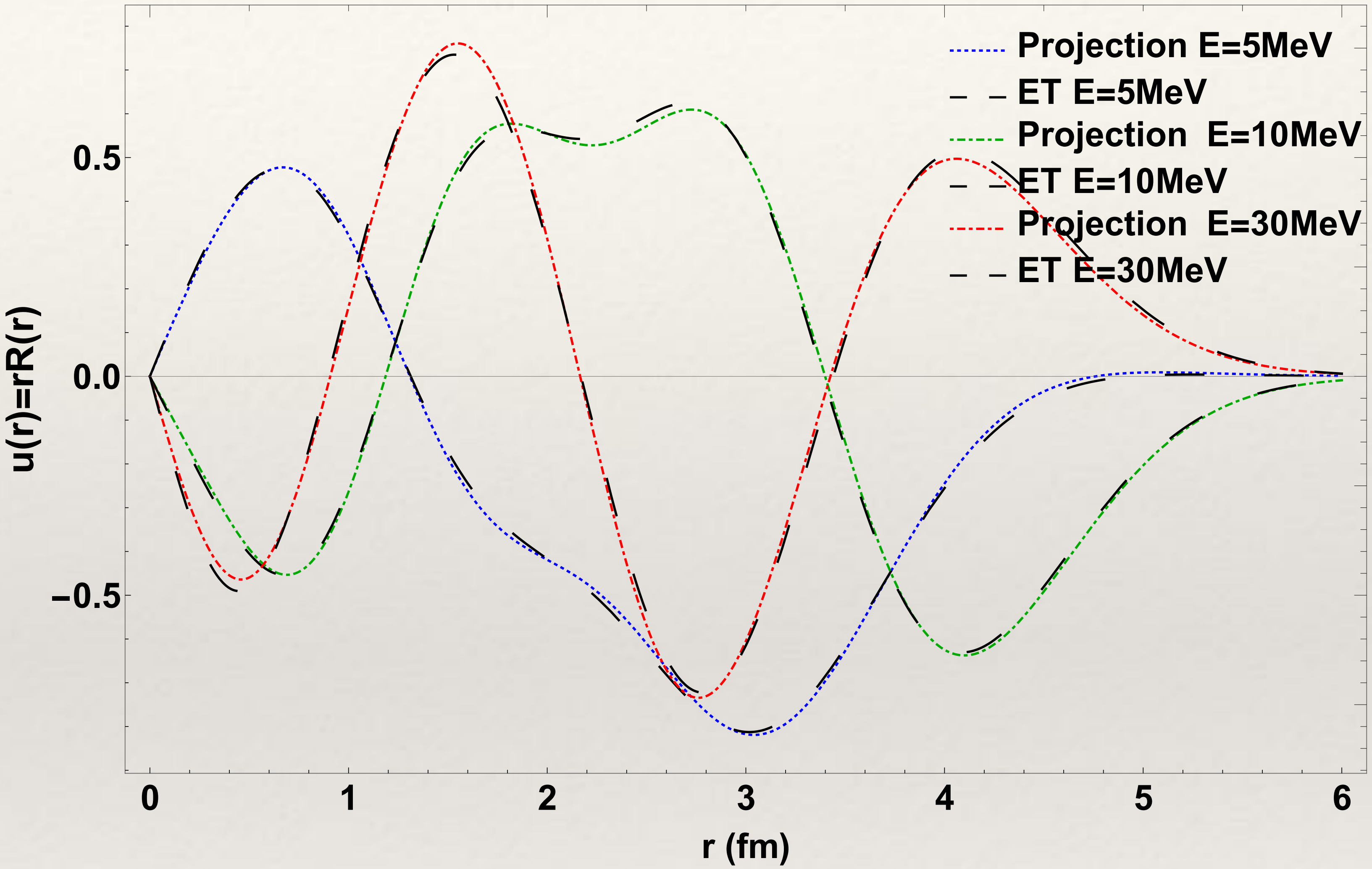
S-Channel Eigenvalue Convergence



Test potential : hard core + well

P Channel Wave Function

- ET Wave functions should match projections of numerical solutions.



- Colored lines are the projections of numerical solutions. Black dashed lines are the effective theory solutions at the same energies.

Operator Renormalization

- Operators can also be matched to an expansion.

$$\hat{O}^{eff}_{ji} = P\frac{E_j}{E_j - HQ}\hat{O}\frac{E_i}{E_i - QH}P$$

$$= P\frac{E_j}{E_j - TQ}\left[\hat{O} + VQ\frac{E_j}{E_j - HQ}\hat{O} + \hat{O}\frac{E_i}{E_i - QH}QV + VQ\frac{E_j}{E_j - HQ}\hat{O}\frac{E_i}{E_i - QH}QV\right]\frac{E_i}{E_i - QT}P$$

$$\rightarrow P\frac{E_j}{E_j - TQ}\left[\hat{O} + \hat{O}_\delta\right]\frac{E_i}{E_i - QT}P$$

- O_δ will have an expansion much like V_δ with an expansion in harmonic oscillator quanta.
- Renormalizing the operator $\hat{1}$ enables recovery of the projected state normalization!
  - An effective Hamiltonian in a P space may reproduce the spectrum, but if you don't know how much of the wave function is represented in P, operator evaluation is suspect.

A-Body ET Calculations

$$H^{eff} = P\left[H\frac{E}{E-QH}\right]P = P\left[\left(\sum_{a\in pairs} H_a\right)\frac{E}{E-QH}\right]P$$

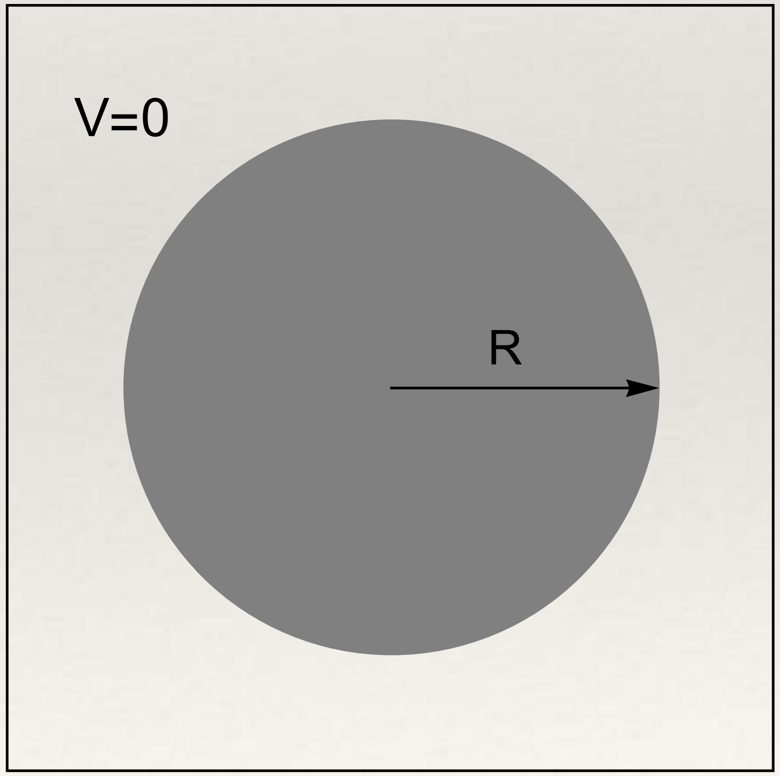
- Expanding and organizing yields

$$H^{eff} = P\left[\sum_a H_a^{eff} + \frac{1}{E}\sum_{a\neq b} H_a^{eff} Q H_b^{eff} + \frac{1}{E^2}\sum_{a\neq b, b\neq c} H_a^{eff} Q H_b^{eff} Q H_c^{eff} + \cdots\right]P$$

- H_a^{eff} is a spectator quanta dependent form of the effective interaction constructed previously.
- This expansion generalizes the effective interaction into an A-body effective Hamiltonian.

Interactions from LQCD

- Lüscher's method can be used to map the spectrum of two nucleons to phase shifts.
  - Use traditional path: collect enough phase shift data in multiple channels and use to fit an effective theory or a model like a realistic potential.
- HAL QCD potential method, Doi *et al.* arXiv:1702.01600
  - Construct Nambu-Bethe-Salpeter wave function and infer non-local potential.
- Sources of error
  - Both: Tail of interaction exceeding L/2.
  - Lüscher's method: Divergences of zeta function in higher order terms.
  - HAL QCD potential: non-elastic excited state contamination.



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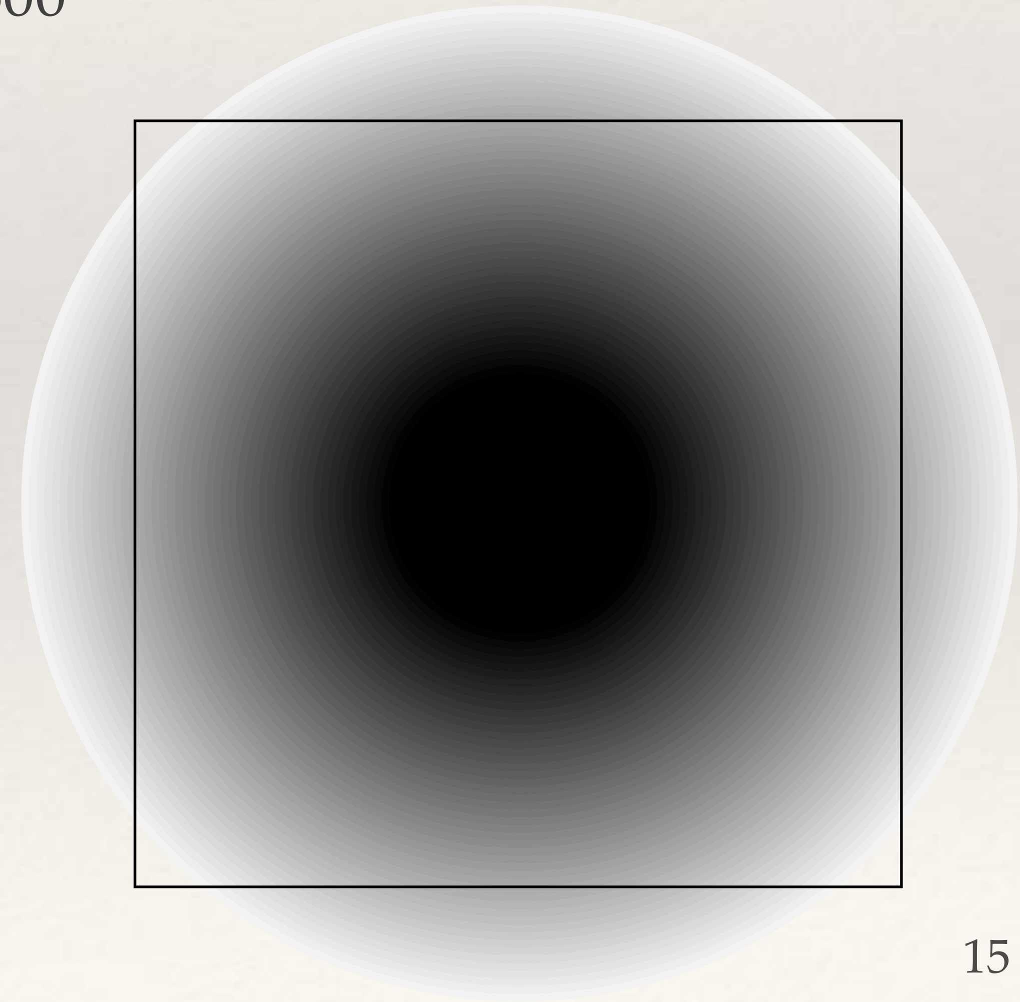

Change: Boundary Conditions

- Phase shifts as boundary conditions are replaced by periodic boundary conditions.
- Small volumes limit the number of states in energy range of interest.
- ET construction should support
  - Multiple volumes to access more states.
  - Boosting

Periodic Momentum Basis

- Even and odd basis functions

$$\phi_{i,s,m}(x)=\sqrt{2/L_i}\sin\left(\alpha_{i,m}x\right),\quad m=1,\ldots,N/2$$

$$\phi_{i,c,0}(x)=\sqrt{2/L_i}(1/\sqrt{2}),\quad m=0$$

$$\phi_{i,c,m}(x)=\sqrt{2/L_i}\cos\left(\alpha_{i,m}x\right),\quad m=1,\ldots,N/2$$

$$\text{with }\alpha_{i,m_i}=2\pi\left|m_i\right|/L_i$$

- *m* ranges from -*N*/2 to *N*/2 with *m*<0 indicating *sin* basis functions

$$\phi_{\vec{m}}(x,y,z)=\phi_{m_x}(x)\,\phi_{m_y}(y)\,\phi_{m_z}(z)$$

- The kinetic energy operator is a bit complicated by the varying side lengths:

$$\hat{T}\phi_{\vec{m}}(x,y,z)=2\pi^2\left(\sum_i\frac{m_i^2}{L_i^2}\right)\phi_{\vec{m}}$$

$$=\lambda_{\vec{m}}\phi_{\vec{m}}(x,y,z)$$

Green's Function for E/(E-QT)

- As before: $\left|\tilde{i}\right\rangle = \frac{E}{E-QT}\left|i\right\rangle = b_{ij}\frac{E}{E-T}\left|j\right\rangle, \quad b_{ij} = \left\{P\frac{E}{E-T}P\right\}_{ij}^{-1}, \quad i,j \in P$
- E/(E-T) is expressed as a bilinear eigenfunction expansion over the periodic basis functions.

$$G_T\left(E;\mathbf{r},\mathbf{r}'\right) = \sum_{\vec{m}} \frac{E}{E-\lambda_{\vec{m}}+i\varepsilon}\phi_{\vec{m}}\left(\mathbf{r}\right)\phi_{\vec{m}}\left(\mathbf{r}'\right)$$

$$b_{\vec{n}'\vec{n}} = \left\langle \vec{n}'\middle|G_T\middle|\vec{n}\right\rangle = \sum_{\vec{m}}\frac{E}{E-\lambda_{\vec{m}}}\left\langle \vec{n}'\middle|\phi_{\vec{m}}\left(\vec{r}'\right)\phi_{\vec{m}}\left(\vec{r}\right)\middle|\vec{n}\right\rangle = \sum_{\vec{m}}\frac{E}{E-\lambda_{\vec{m}}}\chi_{\vec{n}'\vec{m}}\chi_{\vec{n}\vec{m}}$$

$$\textit{where} \quad \chi_{\vec{n},\vec{m}} = \chi_{n_x,m_x}\chi_{n_y,m_y}\chi_{n_z,m_z}, \qquad \chi_{n,m} = \int_{-\infty}^{\infty} dx\, H_n\left(x\right)\phi_m\left(x\right)$$

Green's Function for E/(E-QT)

- As before: $\left|\tilde{i}\right\rangle = \frac{E}{E-QT}\left|i\right\rangle = b_{ij}\frac{E}{E-T}\left|j\right\rangle, \quad b_{ij} = \left\{P\frac{E}{E-T}P\right\}_{ij}^{-1}, \quad i,j \in P$
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3D basis overlap Calculated on the fly	1D basis overlap Stored

Evaluate by Inserting Periodic Basis

Sum T to all orders: $\left\langle \vec{n}' \middle| \frac{E}{E-TQ}\left[T + T\frac{Q}{E}T\right]\frac{E}{E-QT}P \middle| \vec{n} \right\rangle = E\left(\delta_{\vec{n}'\vec{n}} - b_{\vec{n}'\vec{n}}\right)$

- V_{IR} matrix elements are the most expensive part of H_{eff}

$$\left\langle \vec{n}' \middle| G_{TQ} V_{IR} G_{QT} \middle| \vec{n} \right\rangle = \sum_{\vec{m}',\vec{m},\vec{s},\vec{t}} b_{\vec{n}',\vec{s}} \frac{E}{E-\lambda_{\vec{m}'}} \langle \vec{s} | \vec{m}' \rangle \langle \vec{m}' | V_{IR} | \vec{m} \rangle \langle \vec{m} | \vec{t} \rangle \frac{E}{E-\lambda_{\vec{m}}} b_{\vec{t},\vec{n}}$$

- All pieces are precomputed, but sum is still very large.
- For $\vec{n}',\vec{n} \in P^-$ $G_{QT}=1$, which can be used to check results.

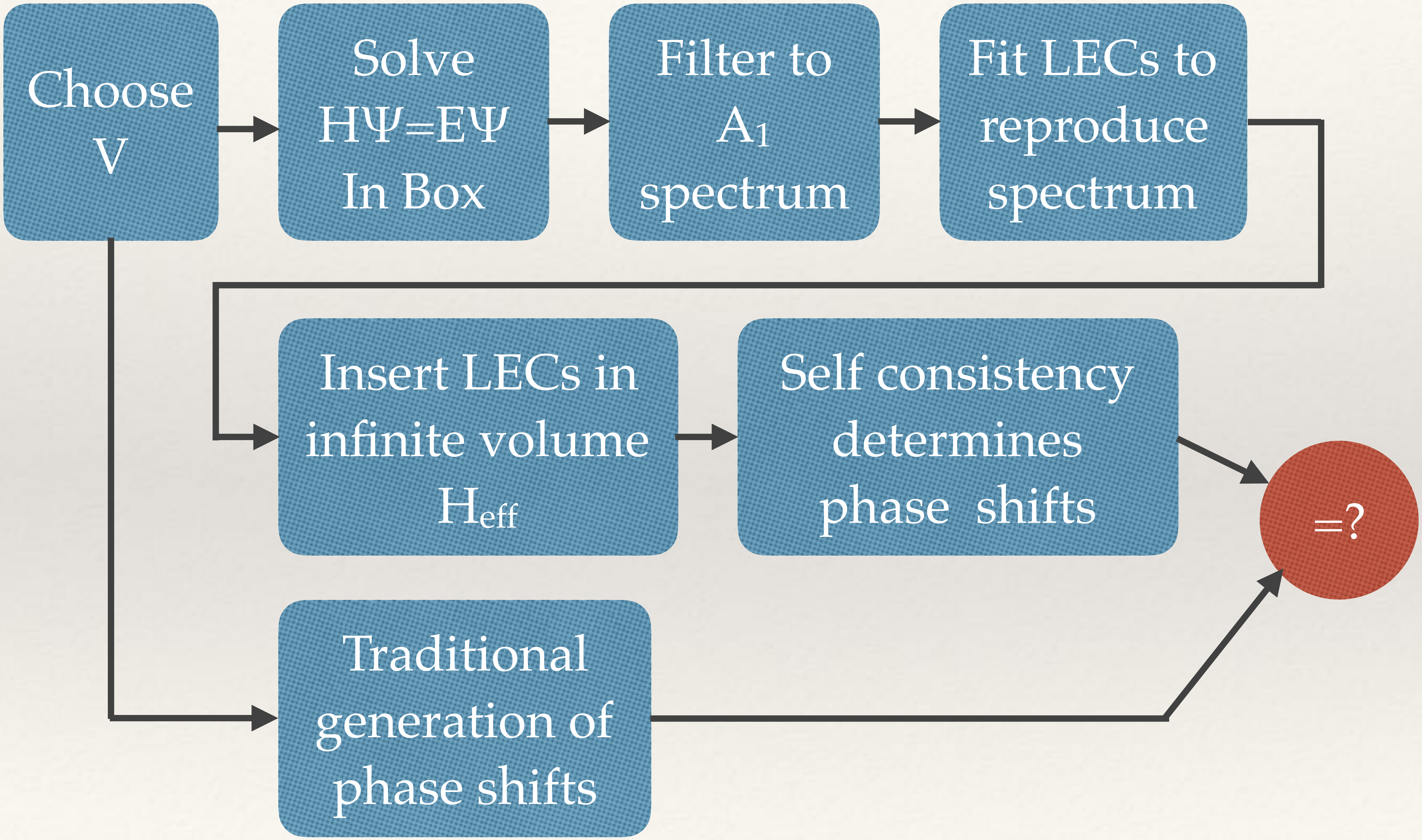
Magic with V_δ

- As long as V_δ on P^- doesn't interact with the boundary it is the same object in both finite and infinite volume contexts.
- Spherical and Cartesian HO bases are simply representations related by brackets.
- Cartesian ET expansion respecting parity invariance.

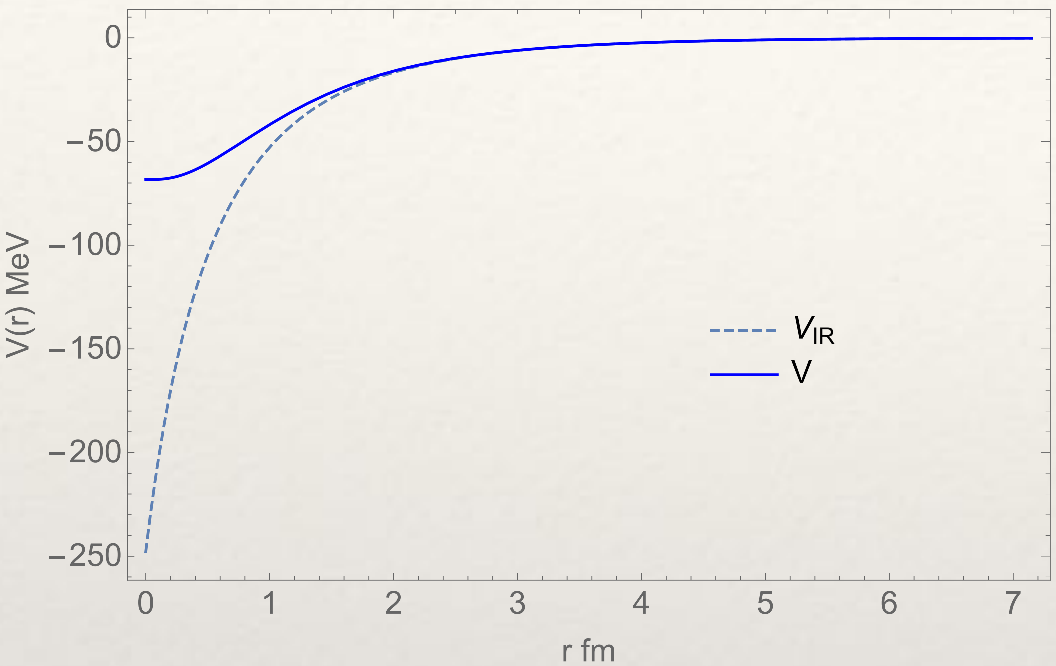
Table 9.1: LECs and Cartesian operators

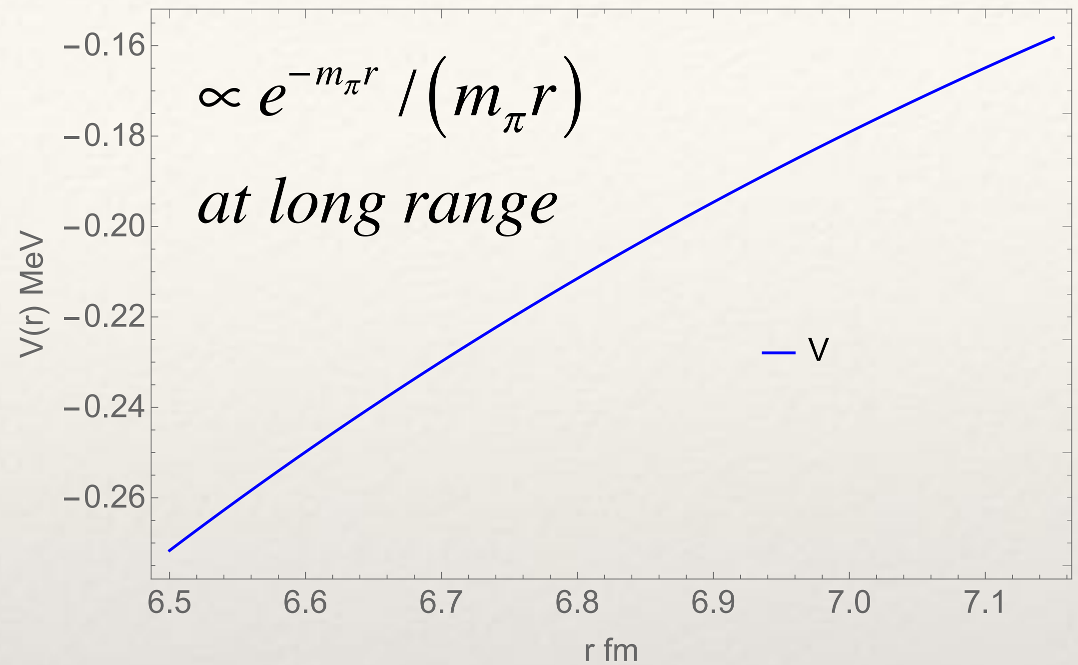
LEC	operators
$c000d000$	$\delta(r)$
$c100d100$	$\left(a_x^\dagger\delta(r)a_x + a_y^\dagger\delta(r)a_y + a_z^\dagger\delta(r)a_z\right)$
$c100d010$	$\left(a_x^\dagger\delta(r)a_y + a_x^\dagger\delta(r)a_z + a_y^\dagger\delta(r)a_z\right) + \mathrm{h.c.}$
$c200d000$	$\left(a_x^{\dagger 2} + a_y^{\dagger 2} + a_z^{\dagger 2}\right)\delta(r) + \mathrm{h.c.}$
$c110d000$	$\left(a_x^\dagger a_y^\dagger + a_x^\dagger a_z^\dagger + a_y^\dagger a_z^\dagger\right)\delta(r) + \mathrm{h.c.}$

Testing Plan



Test Setup: Range(V)>L/2





$L = 14.3\ fm$

$m_\pi L = 10$

- Periodic images of the potential make a contribution.
- Infinite volume bound state at -4.05 MeV.
- LECs are fit using states with L=0 overlap.

Rep	MeV	L=0	L=2	L=4	L=6
A_1^+	-4.4428	0.5	0	0.866	0
A_1^+	2.0314	0.155	0	0.988	0
E^+	7.5995	0	0.424	0.361	0.830
E^+	15.2980	0	0.474	0.393	0.788
A_1^+	21.6167	0.326	0	0.265	0.908
E^+	23.2423	0	0.468	0.597	0.651
A_1^+	29.4041	0.521	0	0.853	0.023
E^+	30.9457	0	0.567	0.428	0.704
A_1^+	35.2449	0.655	0	0.189	0.732
E^+	38.4043	0	0.882	0.176	0.437
A_1^+	45.1402	0.526	0	0.576	0.625

Phase Shift Comparison Setup

- Reference phase shifts for L=0 and L=4 are directly calculated from V.
- HOBET S-channel phase shifts are computed from the N3LO LECs that reproduce the spectrum. The phase shift is found by dialing the phase shift to produce energy self consistency.
- Lüscher's method phase shifts come from the formula

$$k\cot\delta_0 = \frac{2}{\sqrt{\pi}L}\mathcal{Z}_{0,0}\left(1;\tilde{k}^2\right) + \frac{12288\pi^7}{7L^{10}}\frac{\mathcal{Z}_{4,0}\left(1;\tilde{k}^2\right)^2}{k^9\cot\delta_4} + \mathcal{O}\left(\tan^2\delta_4\right)$$

Luu, Savage, arXiv:1101.3347

- An effective range expansion up to k^6 is used to interpolate.
- For simplicity the second term is evaluated using the L=4 phase shift directly determined from V.

Phase Shift Results

$L = 14.3\ fm$

$m_\pi L = 10$

The V column should be considered the reference.

E MeV	V	HOBET	Leading Lüscher	Next Order Lüscher
1	142.023	141.931	142.552	142.751
2	128.972	128.860	129.571	129.823
4	113.602	113.464	114.205	114.403
8	96.919	96.752	97.575	97.3135
10	91.473	91.296	92.228	91.6403
15	81.672	81.480	82.852	81.3184
20	74.876	74.691	76.667	74.0936

- A potential source of error for both HOBET and Lüscher's method is the accuracy of the finite volume spectrum.
  - Solved three times with N=350,400,450 and made a continuum extrapolation. The 3 results showed a consistent and small evolution of the eigenvalues.

Summary

- The HOBET interaction can be directly constructed from observables such as phase shifts in the continuum.
- Energy dependence is a virtue, enabling a complete sum of kinetic energy scattering, isolating short range physics for the ET expansion.
- The interaction can be used in an A-body context with the excitation of spectators determining Λ for the interaction.
- Operator renormalization is natural, including correct normalization of states - one simply renormalizes the "1" operator.
- The same ET expansion is valid in a periodic volume, enabling matching to the LQCD spectrum with the same LEC values as in the infinite volume case!

Thanks to my collaborator on this project - Wick Haxton

End