Lattice QCD and the Proton Radius

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(LHP collaboration)

[1711.11385; Phys.Rev.D97.034504]

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Outline

- Nucleon form factors and radii on a lattice
- Results from the physical point
- Form factors at zero momentum

Basics of Hadron Structure in Lattice QCD

Lattice Field Theory ⇔ Numerical evaluation of the Path Integral

$$\langle q_x \bar{q}_y \ldots \rangle = \int \mathcal{D}\big(Glue\big) \int \mathcal{D}\big(Quarks\big)\; e^{-S_{Glue} - \bar{q}\left(\not{D}+m\right)q} \;\big[q_x \bar{q}_y \ldots\big]$$

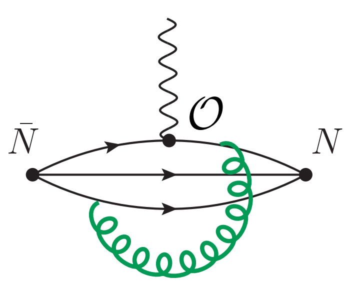
Grassmann integration

$$= \underbrace{\int \mathcal{D}\big(Glue\big)\; e^{-S_{Glue}}\; \mathrm{Det}\big(\not{D}+m\big)}_{\text{Hybrid Monte Carlo}} \;\Big[\big(\not{D}+m\big)^{-1}_{x,y} \ldots\Big]$$

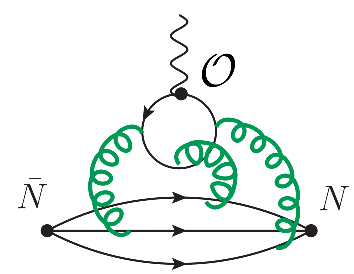
$(\not{D}+m)\cdot q = 0$
quark motion in
gluon background

Hadron Matrix Elements:

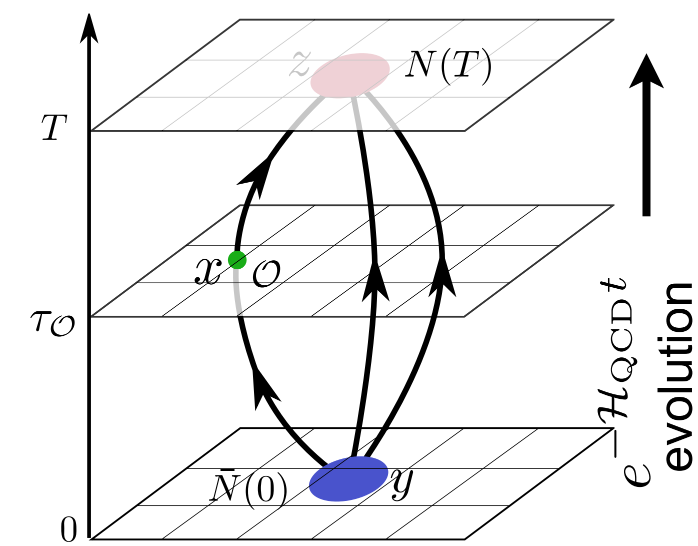
$$C^{\mathcal{O}}_{3\mathrm{pt}}(T) = \langle N(T)\mathcal{O}(\tau)\bar{N}(0)\rangle =$$



"connected"



"disconnected"



Each quark line = $(\not{D}+m)^{-1}\cdot\psi$

$$\langle N(T)\mathcal{O}(\tau)N(0)\rangle = \sum_{n,m} Z_m e^{-E_n(T-\tau)} \langle n|\mathcal{O}|m\rangle e^{-E_m\tau} Z_n^*$$

$$\xrightarrow[T\to\infty]{} Z_{00} e^{-M_N T}\Big[\langle P'|\mathcal{O}|P\rangle + \mathcal{O}\big(\underbrace{e^{-\Delta E_{10}T}, e^{-\Delta E_{10}\tau}, e^{-\Delta E_{10}(T-\tau)}}_{\text{excited states}}\big)\Big]$$

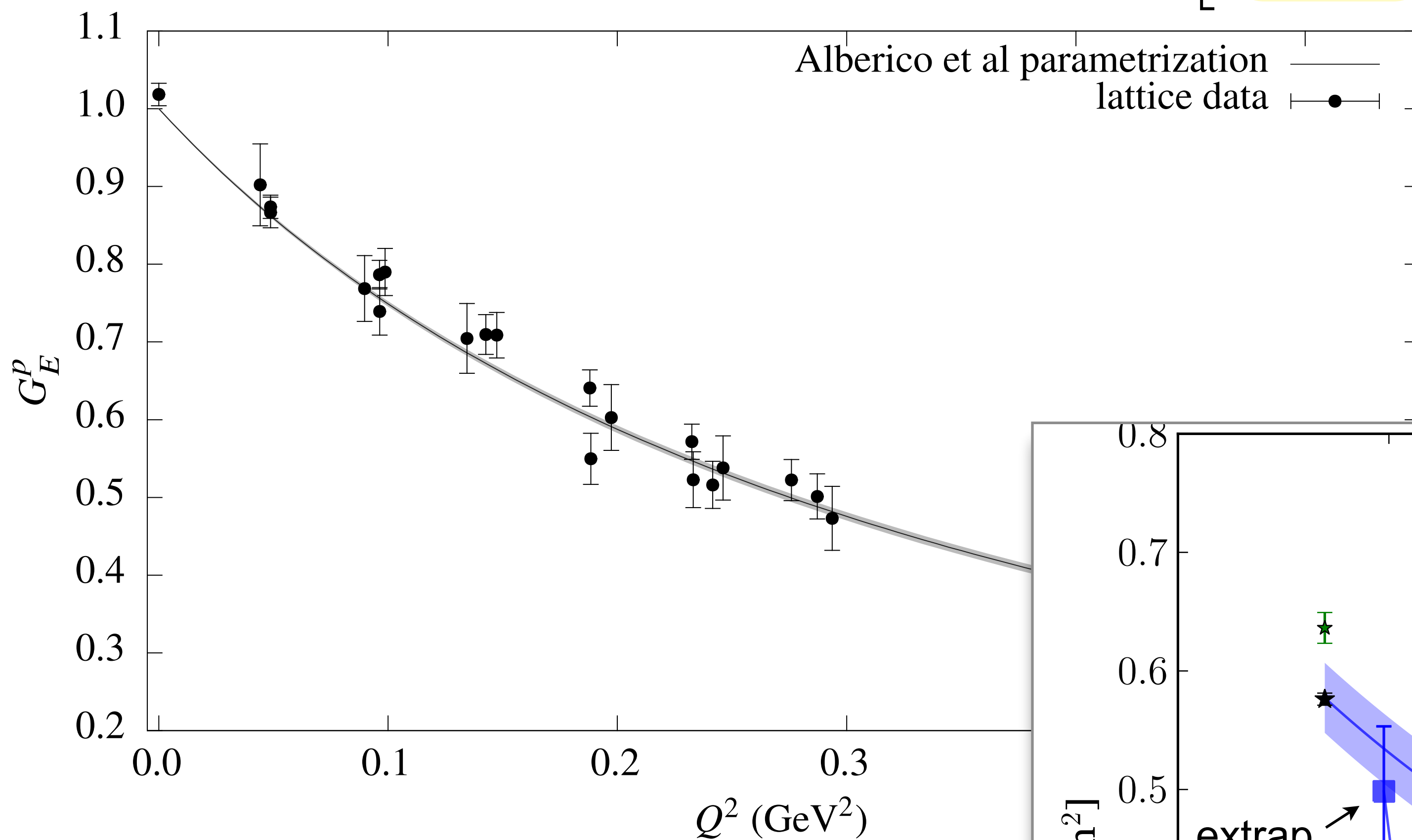
Ground state
form factors

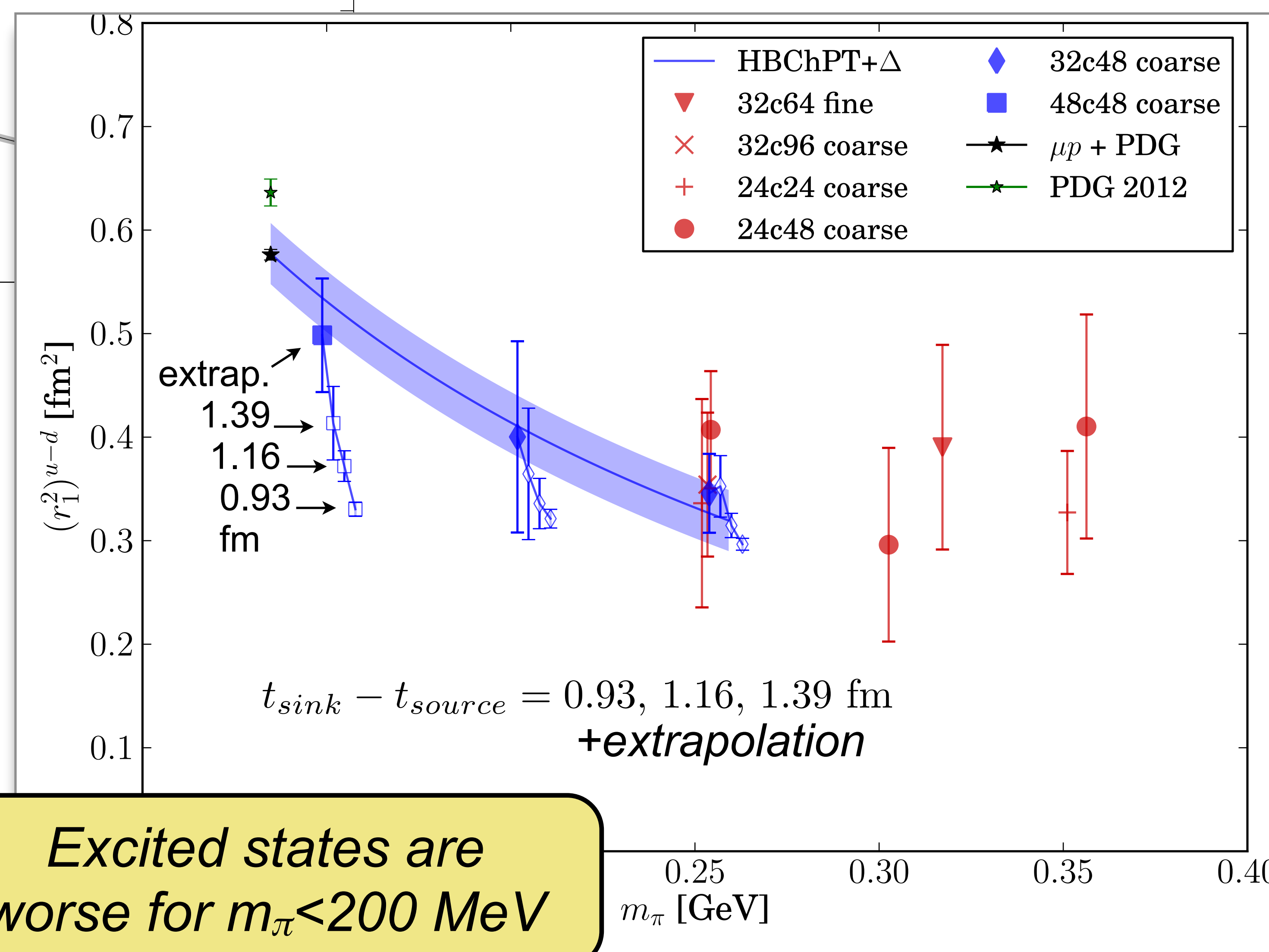
Systematic effects
- excited states
- discretization errors
- finite volume
- ~~unphysical (heavy) pion mass~~

Electric Form Factor

$$\langle P+q|\,\bar{q}\gamma^\mu q\,|P\rangle = \bar{U}_{P+q}\left[F_1(Q^2)\gamma^\mu + F_2(Q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M_N}\right]U_P$$

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2}F_2(Q^2)$$



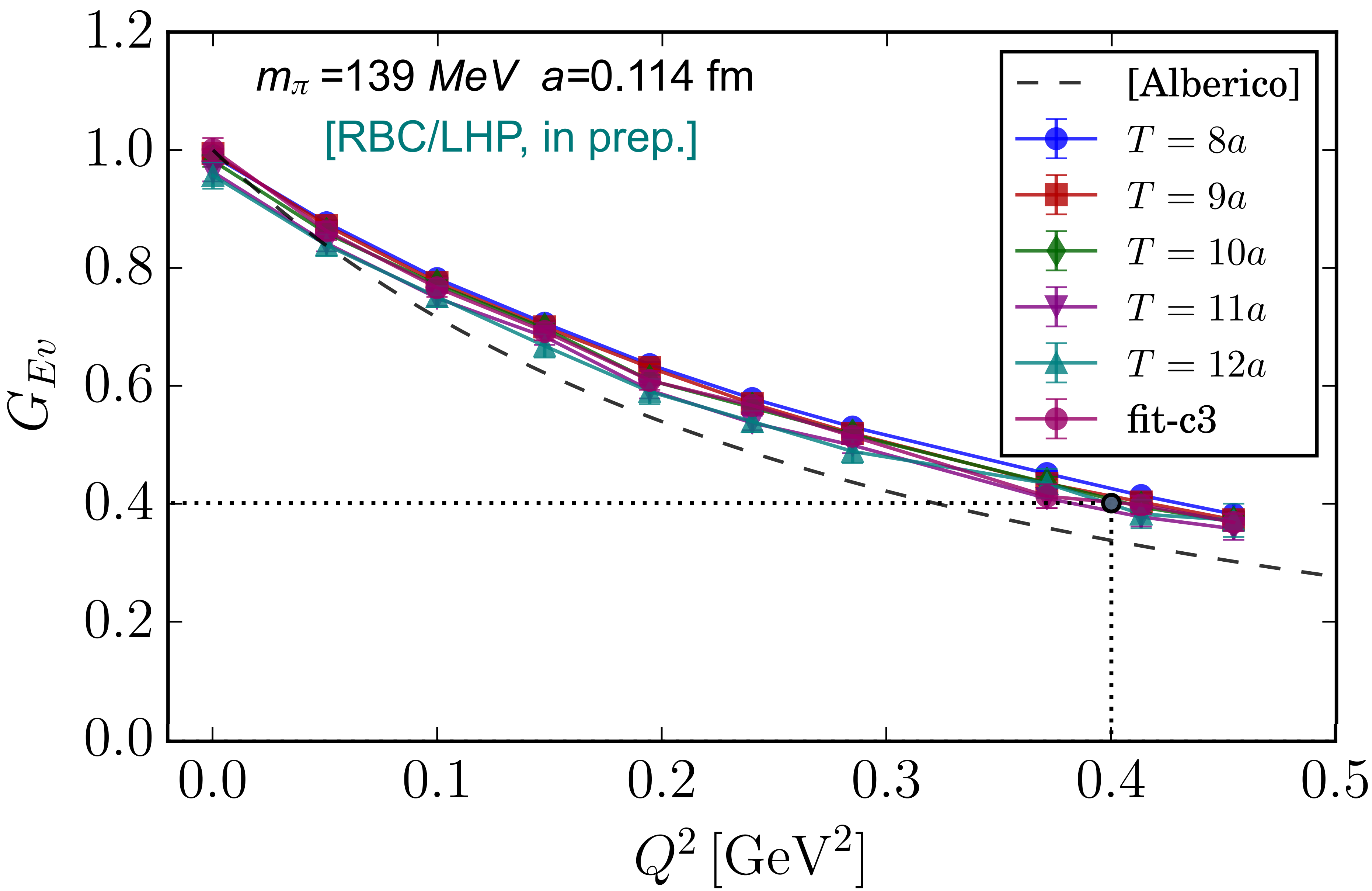


$$G_{Ep}(Q^2) \approx 1 - \frac{1}{6}Q^2\langle r_E^2\rangle^p + O(Q^4)$$

$$\langle r^2\rangle = -6\frac{dG(Q^2)}{dQ^2}\bigg|_{Q^2=0}$$

Excited states are worse for m_π<200 MeV

Electric Form Factor at the Physical Point



$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2} F_2(Q^2)$$

$$G_{Ev} = G_{Ep} - G_{En}$$

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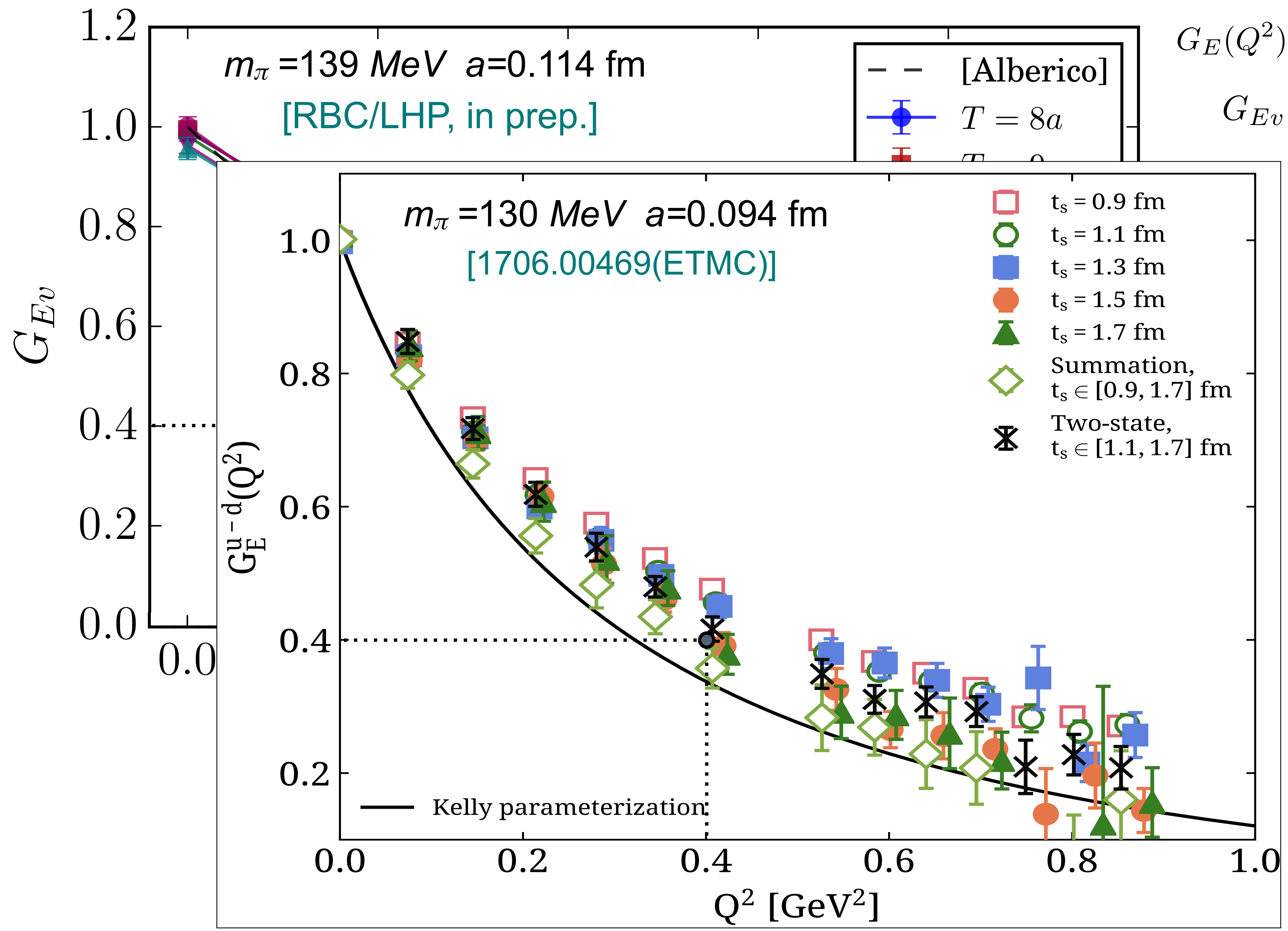

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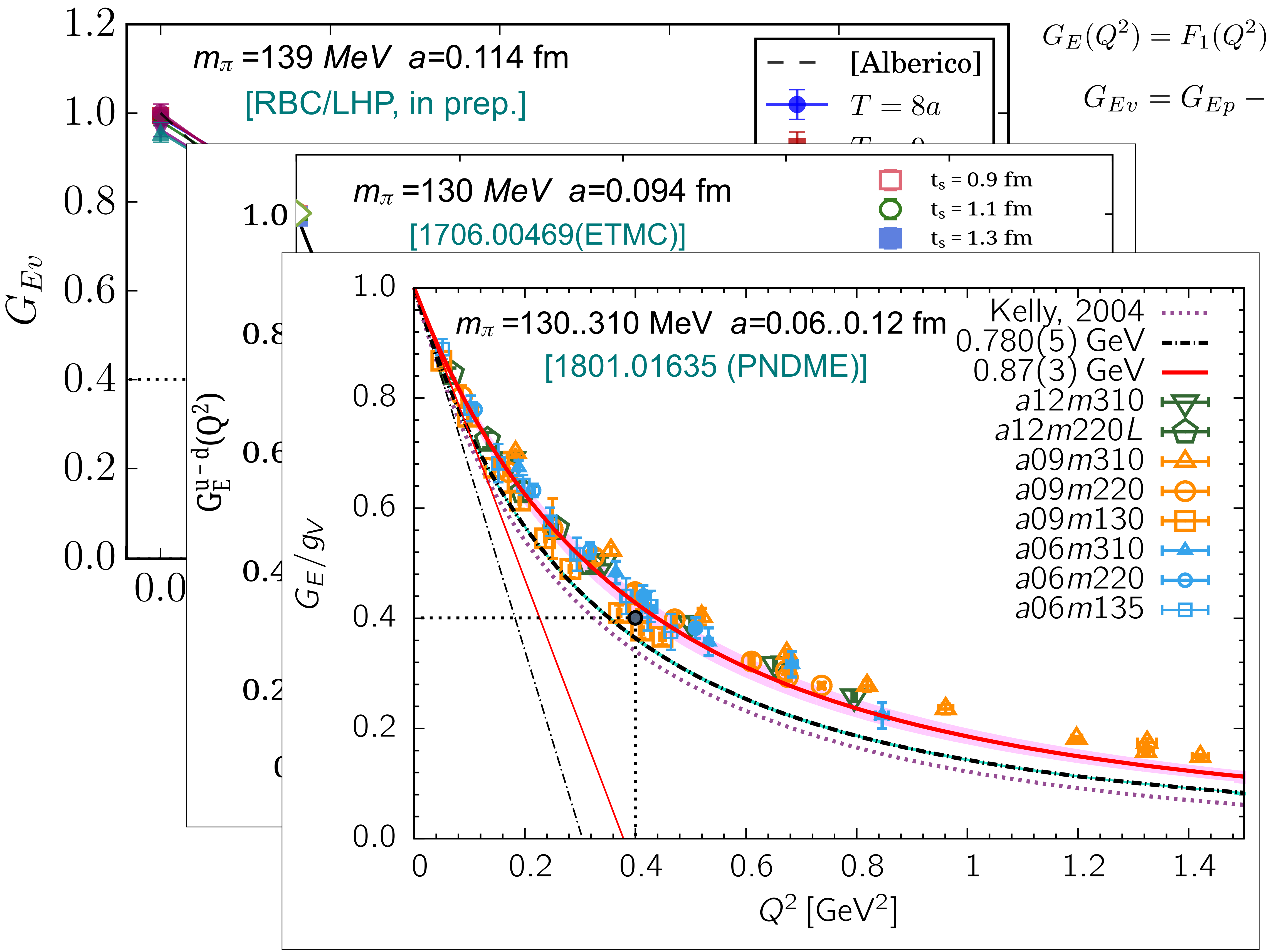

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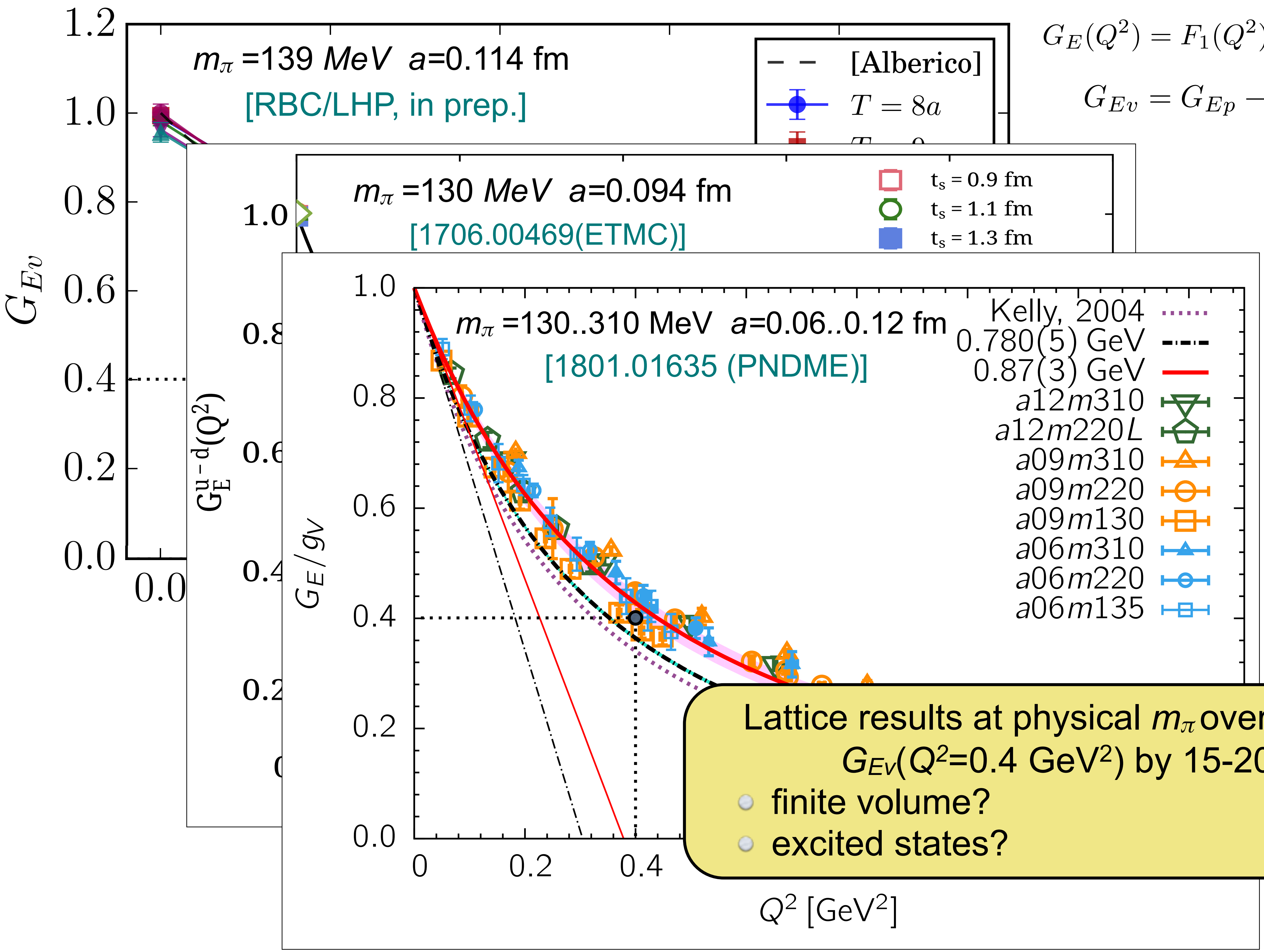


Lattice results at physical m_π overestimate $G_{Ev}(Q^2$=0.4 GeV^2) by 15-20%

- finite volume?
- excited states?

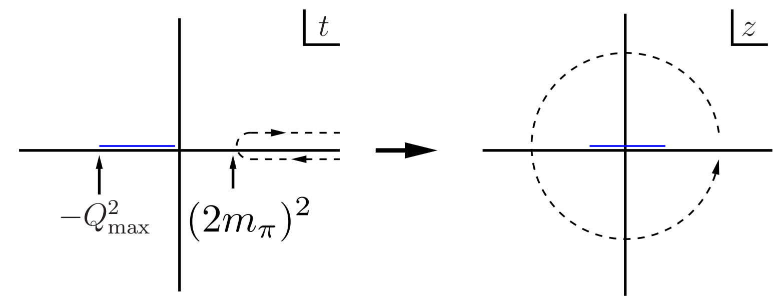
Charge Radius at/near the Physical Pion mass

- "dipole" fits

$$G(Q^2) \sim \frac{1}{(1+Q^2/M_D^2)^2}$$

$$\langle r^2 \rangle = \frac{12}{M_D^2}$$

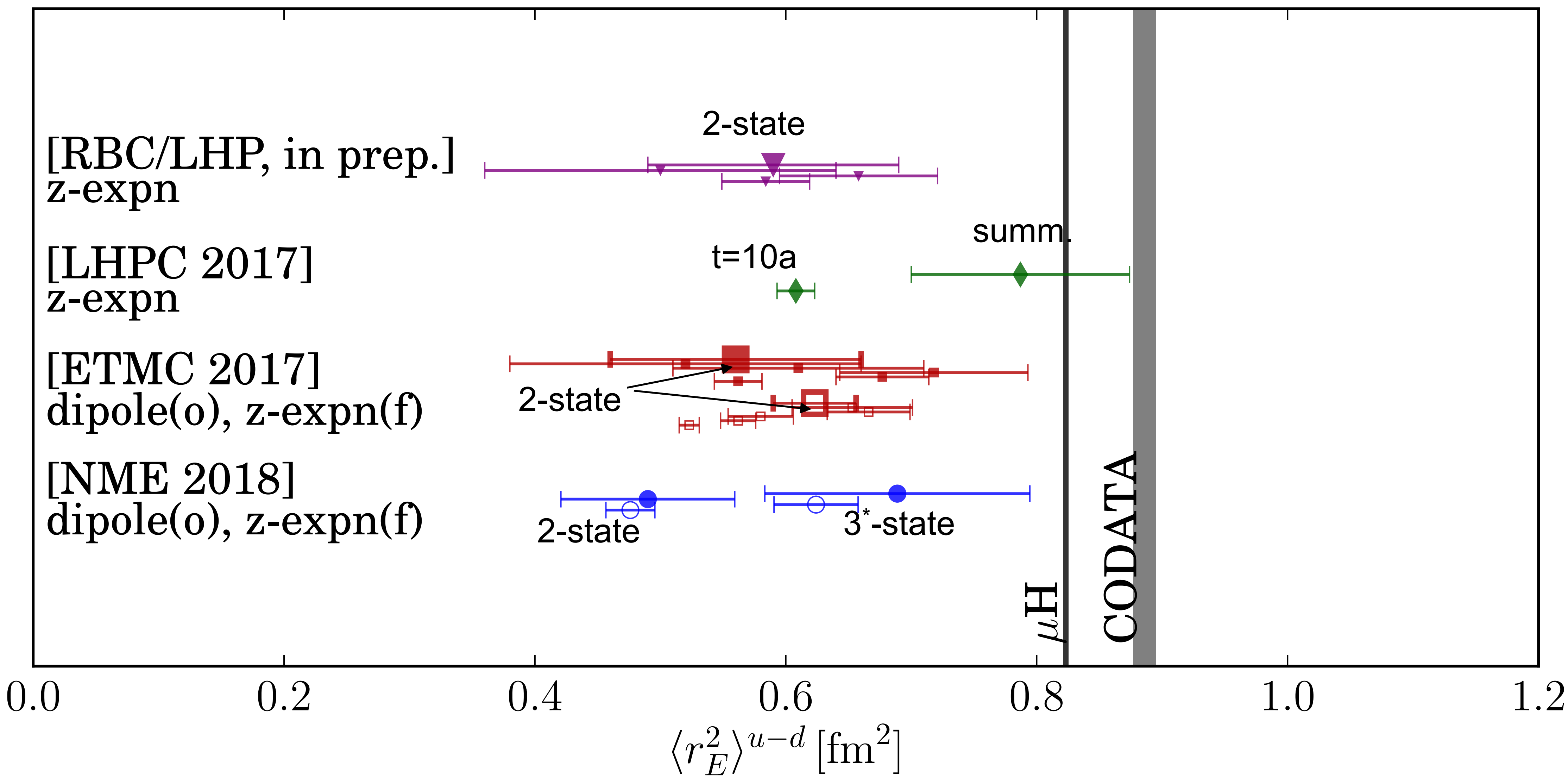
- model-independent fits (eg z-expansion)



$$z = \frac{\sqrt{t_{cut}-t}-\sqrt{t_{cut}-t_0}}{\sqrt{t_{cut}-t}+\sqrt{t_{cut}-t_0}}$$

$$G(Q^2) \sim \sum a_k [z(t=-Q^2)]^k$$

- strong contributions from excited states



Smaller Momenta: Twisted Boundary Conditions

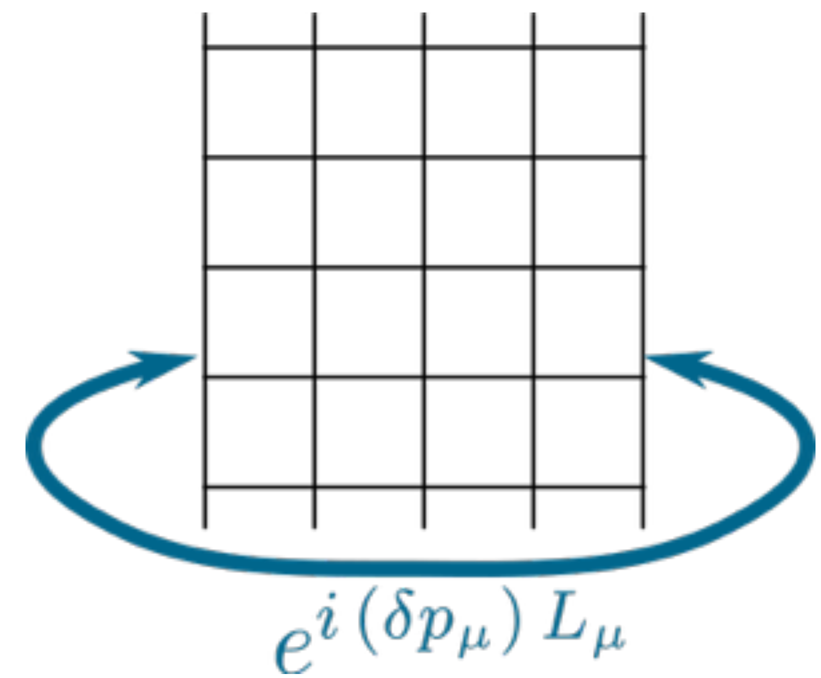
- Quantized lattice momenta for PBC

$$\psi(x+L) = \psi(L) \qquad p_\mu = \frac{2\pi}{L_\mu} n_\mu$$

with minimally acceptable $m_\pi L \gtrsim 4$

$$p_{\min} \lesssim \frac{\pi}{2} m_\pi \approx 0.21\,\mathrm{GeV}$$

$$Q^2_{\min} \approx 0.05\,\mathrm{GeV}^2$$



- Twisted BC $\quad \psi(x+\hat{L}_\mu) = e^{i\theta_\mu}\psi(x)$

arbitrary momenta $\quad p_\mu = \dfrac{2\pi}{L_\mu} n_\mu + \dfrac{\theta_\mu}{L_\mu}$

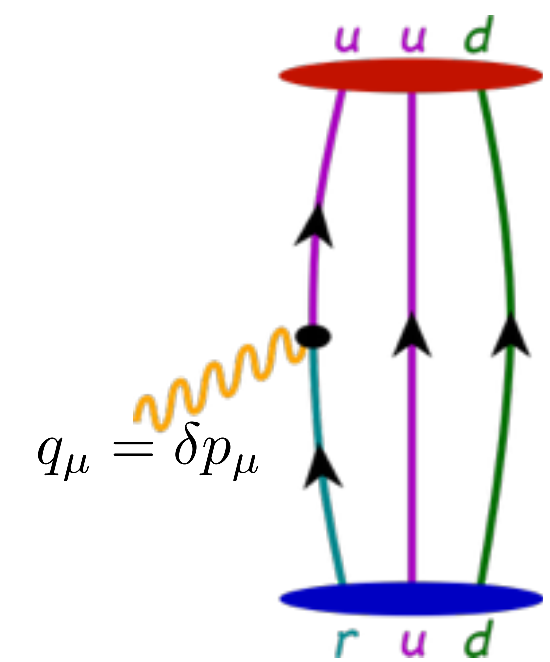
baryons with new twisted valence "flavor" $\;r$

$$\chi_{\Sigma_r} = \frac{1}{\sqrt{2}}\left([rud] + [rdu]\right),$$

$$\chi_{\Lambda_r} = \frac{1}{\sqrt{6}}\left(2[udr] - [rud] - [dru]\right),$$

no sea twisted flavor ⇒ additional finite volume effects

[F.J.Jiang, B.Tiburzi (0810.1495)]

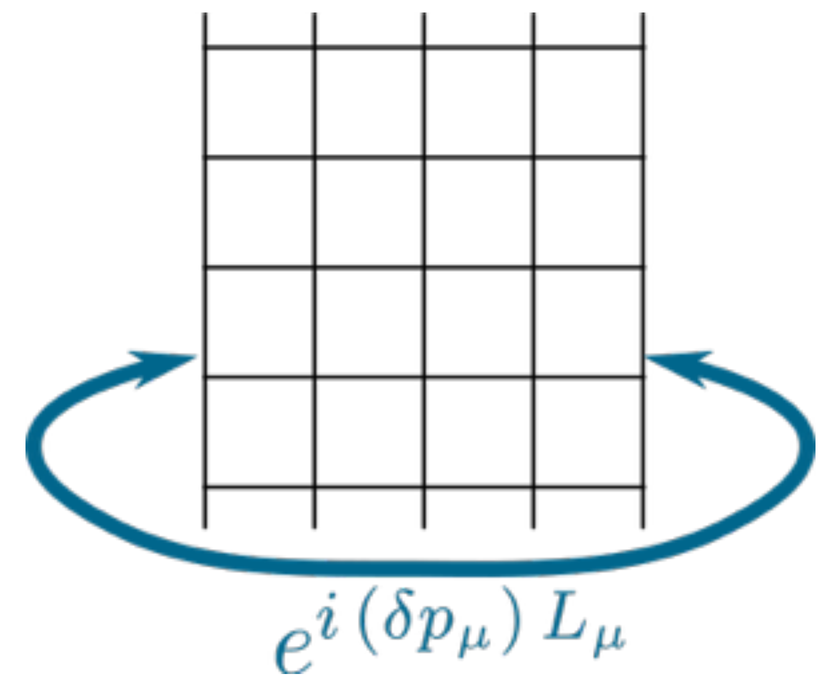


$$\langle N(\vec{p}\,'=0)\, J_\nu(q) \bar{N}(0)\rangle$$

Expansion in Lattice Momentum: Correlators

[de Divitiis, Petronzio, Tantalo (2012)]
compute correlator expansion

$$C(q) = C(0) + \delta p_\mu \frac{\partial C(q)}{\partial q_\mu} + \frac{1}{2}\delta p_\mu \delta p_\nu \frac{\partial^2 C(q)}{\partial q_\mu \partial q_\nu} + \dots$$

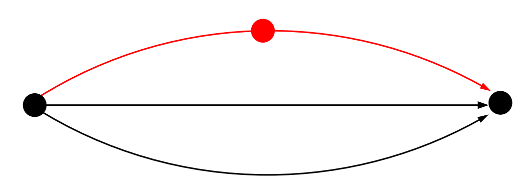


using propagator derivatives

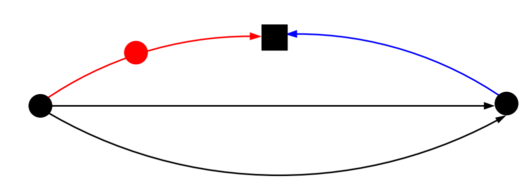
$$\frac{\delta \not{D}^{-1}}{\delta p_\mu} = -\not{D}^{-1} \frac{\delta \not{D}}{\delta p_\mu} \not{D}^{-1}$$

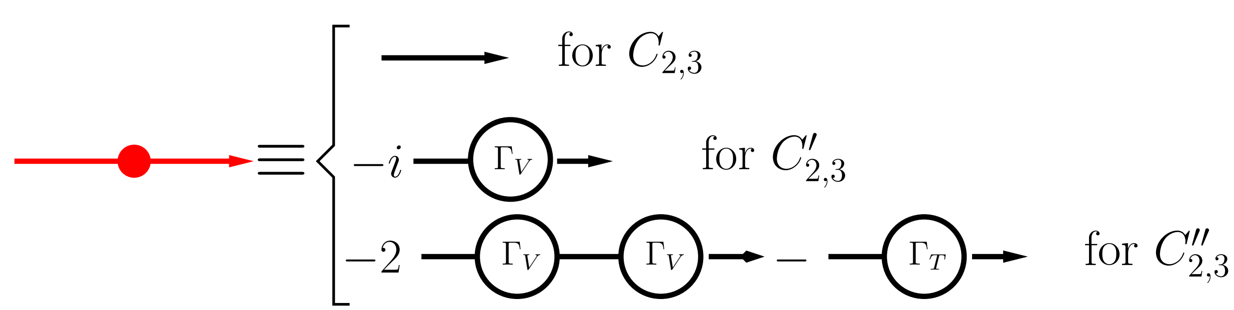
$$\frac{\delta^2 \not{D}^{-1}}{\delta p_\mu^2} = -\not{D}^{-1} \boxed{\frac{\delta^2 \not{D}}{\delta p_\mu^2}} \not{D}^{-1} + 2\not{D}^{-1} \frac{\delta \not{D}}{\delta p_\mu} \not{D}^{-1} \frac{\delta \not{D}}{\delta p_\mu} \not{D}^{-1}$$

"tadpole" **(conserved) vector current**

$C_2^{(n)} =$ 

Implementation on a lattice: compute nucleon correlators with "sequential" propagators

$C_3^{(n)} =$ 



Expansion in Lattice Momentum: Matrix Elements

Estimator for matrix elements

$$R_N^X = \frac{C_3^{\mathscr{O}_X^{q,\mu}}(\vec{p},\vec{p}',\tau,T)}{\sqrt{C_2(\vec{p},T)C_2(\vec{p}',T)}} \qquad R_S = \sqrt{\frac{C_2(\vec{p},T-\tau)C_2(\vec{p}',\tau)}{C_2(\vec{p}',T-\tau)C_2(\vec{p},\tau)}}$$

$$R_X^{q,\mu}(\vec{p},\vec{p}',\tau,T) = R_N^X R_S = M_X^{q,\mu}(\vec{p},\vec{p}') + O(e^{-\Delta E_{10}(\vec{p})\tau}) + O(e^{-\Delta E_{10}(\vec{p}')(T-\tau)}) + O(e^{-\Delta E_{min}T})$$

$$\downarrow$$

$$\frac{\sum_{\lambda,\lambda'}\bar{u}(\vec{p},\lambda)\Gamma_{\rm pol}u(\vec{p}',\lambda')\langle p',\lambda'|\mathscr{O}_X^{q,\mu}|p,\lambda\rangle}{4\sqrt{E(\vec{p})E(\vec{p}')(E(\vec{p})+m)(E(\vec{p}')+m)}}$$

converges to the ground state with T→∞

Vector current insertion

$$R_V^0 = 1, \qquad \partial_1 R_V^2 = -\frac{i}{2m}G_M(0),$$

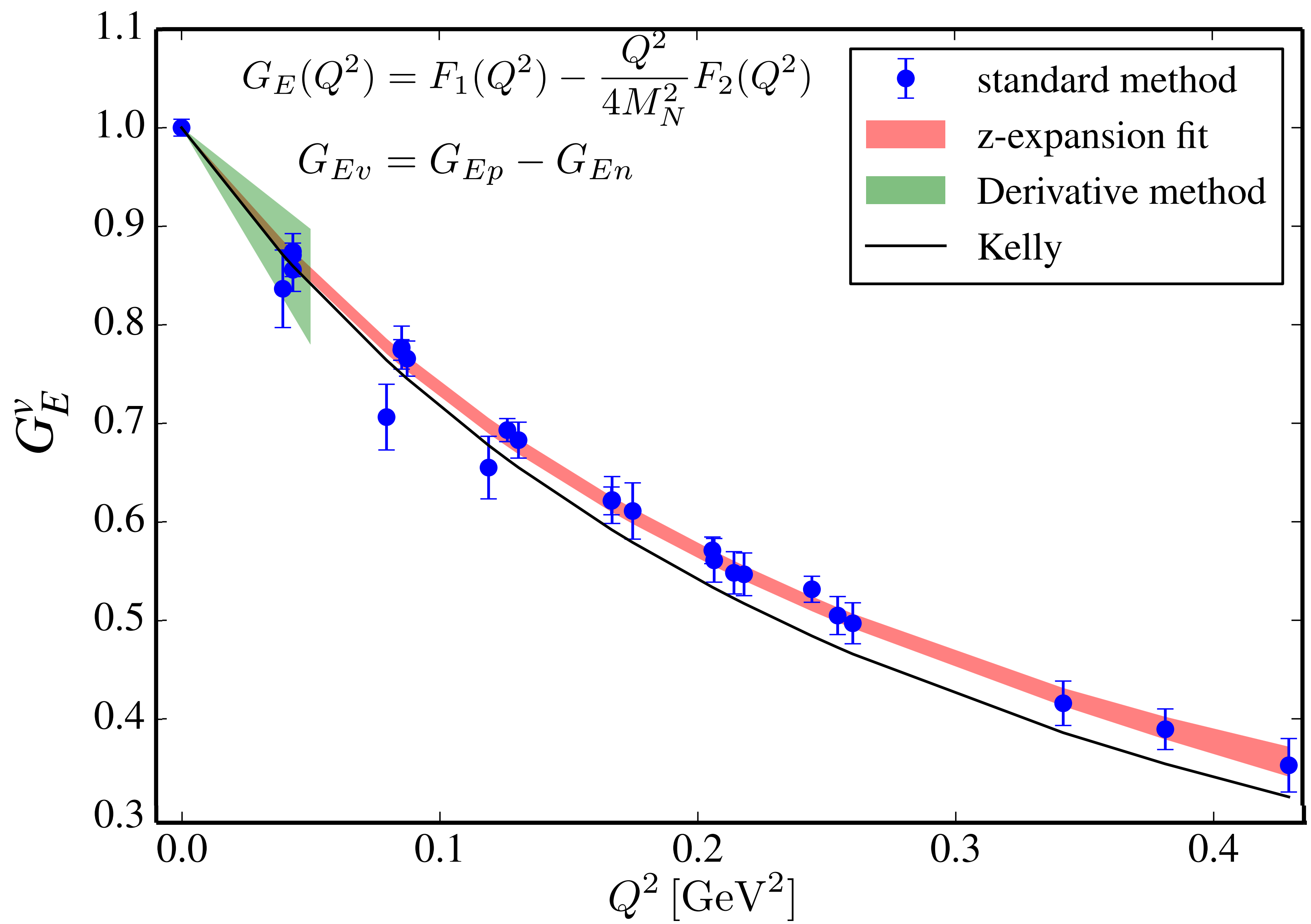
$$\partial_2 R_V^1 = \frac{i}{2m}G_M(0) \qquad \partial^2_{1,2,3}R_V^0 = -\frac{1}{4m^2} - \frac{1}{3}r_E^2,$$

$$\Rightarrow$$

$$\mu = 2i\,m\,(R_V^2)',$$

$$r_E^2 = -\frac{3}{4m^2} - 3\,\frac{(R_V^0)''}{R_V^0}$$

Isovector Electric Form Factor



comparison of G_{Ev} slope at Q_2=0

- z-expansion fit vs.
- twist-derivative method

m_π =135 MeV a=0.093 fm 64^4 (BMWc)

[N.Hasan et al, PRD97: 034504 (1711.11385)]

Charge Radius from FF. at Zero Momentum

- "dipole" fits

$$G(Q^2) \sim \frac{1}{(1+Q^2/M_D^2)^2}$$

$$\langle r^2 \rangle = \frac{12}{M_D^2}$$

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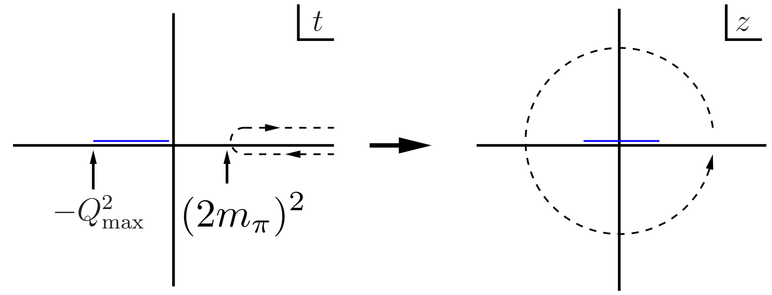

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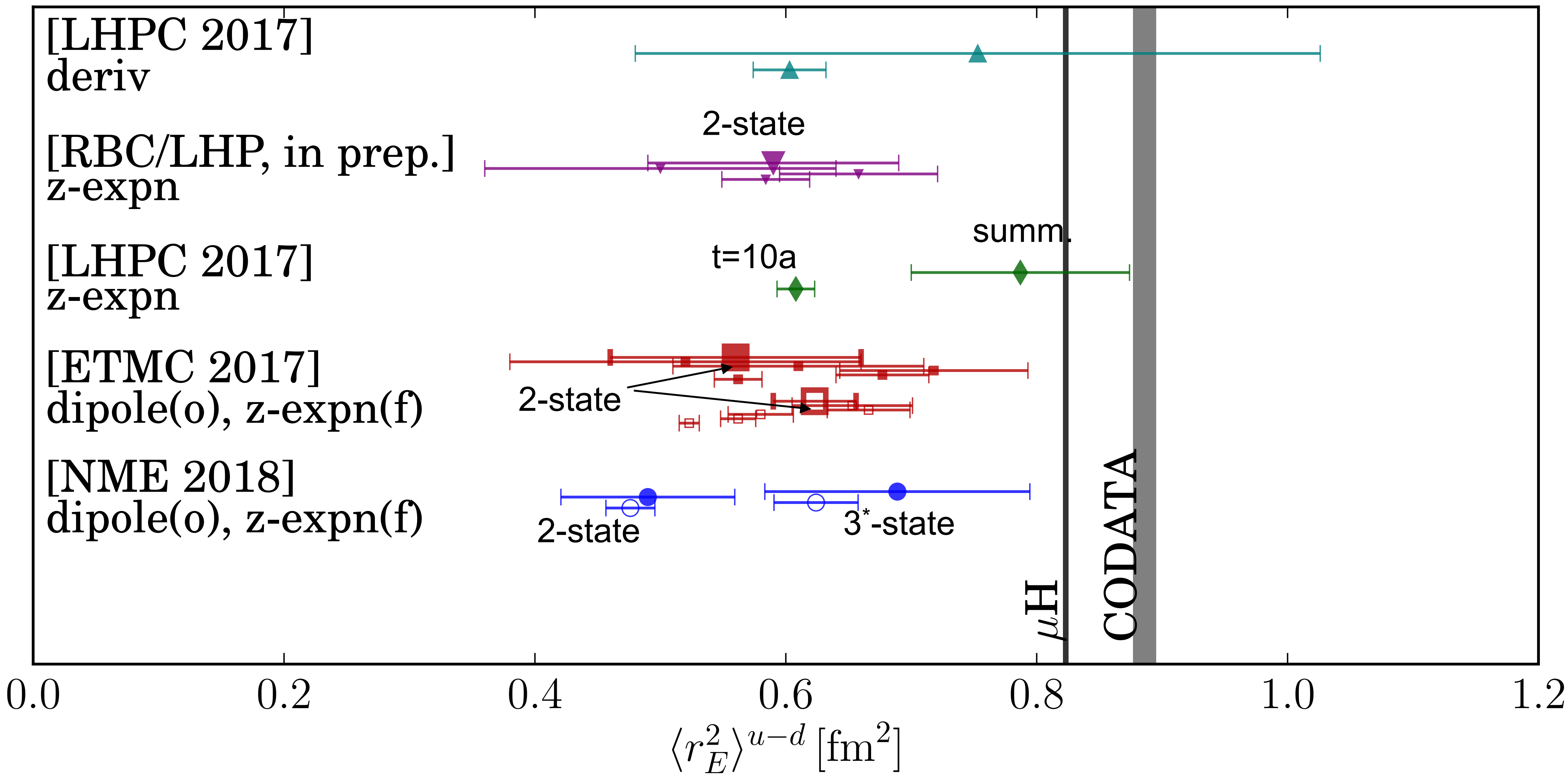


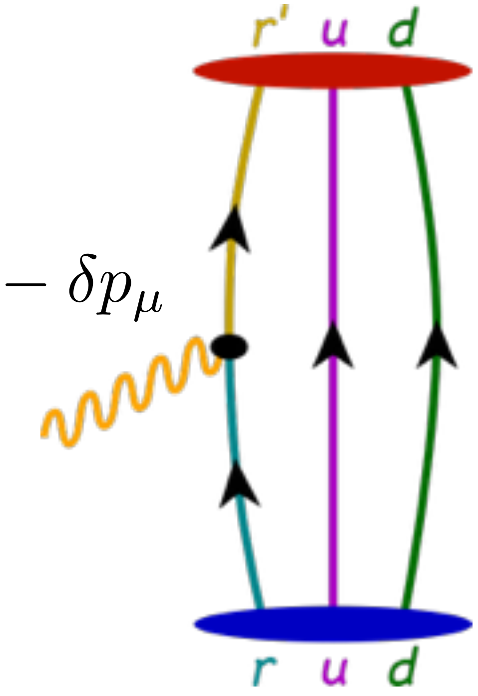
Derivatives wrt. Initial and Final Momenta

Evaluate the radius from varying the initial and final momenta

[B.Tiburzi, 1407.1459] $$\frac{\partial^2}{\partial p'_i \partial p_i}\langle N(p')J^0 N(p)\rangle$$

No tadpole insertions in propagator derivatives

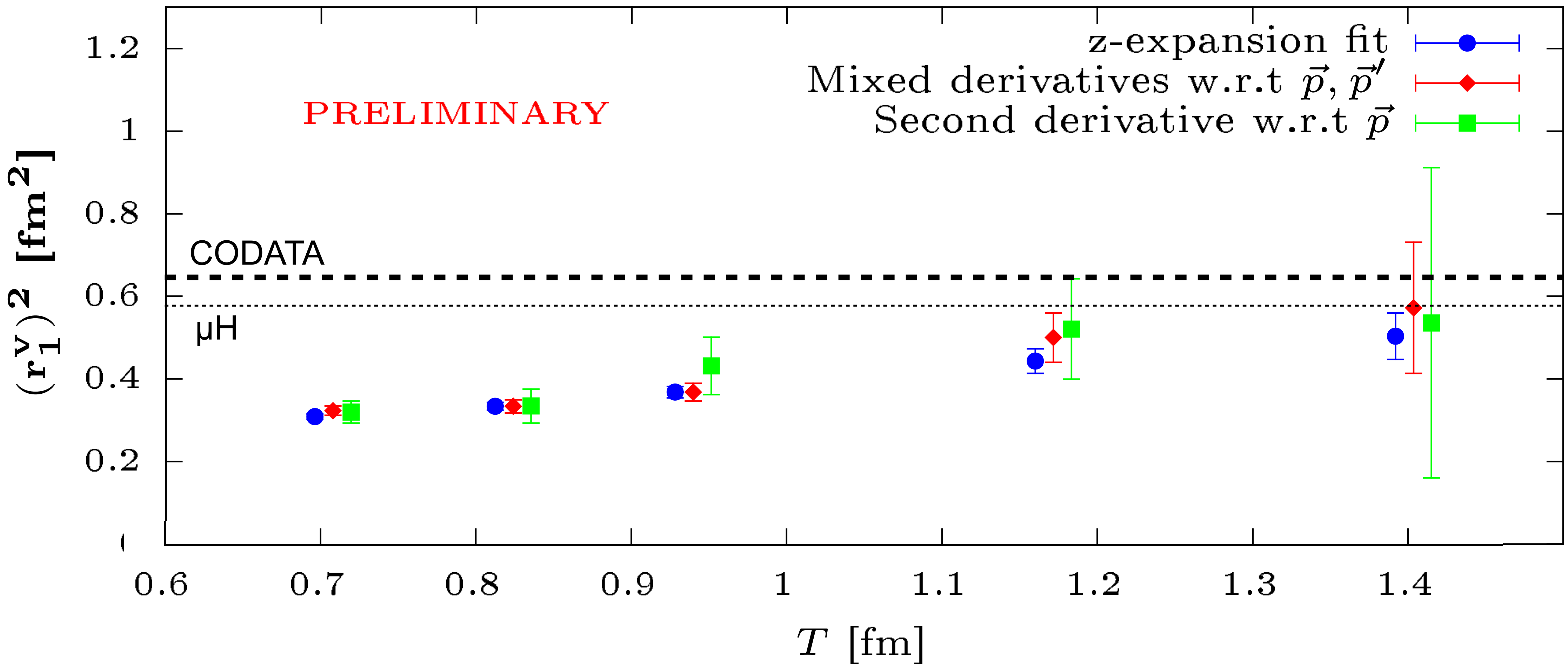
$$q_\mu = \delta p'_\mu - \delta p_\mu$$



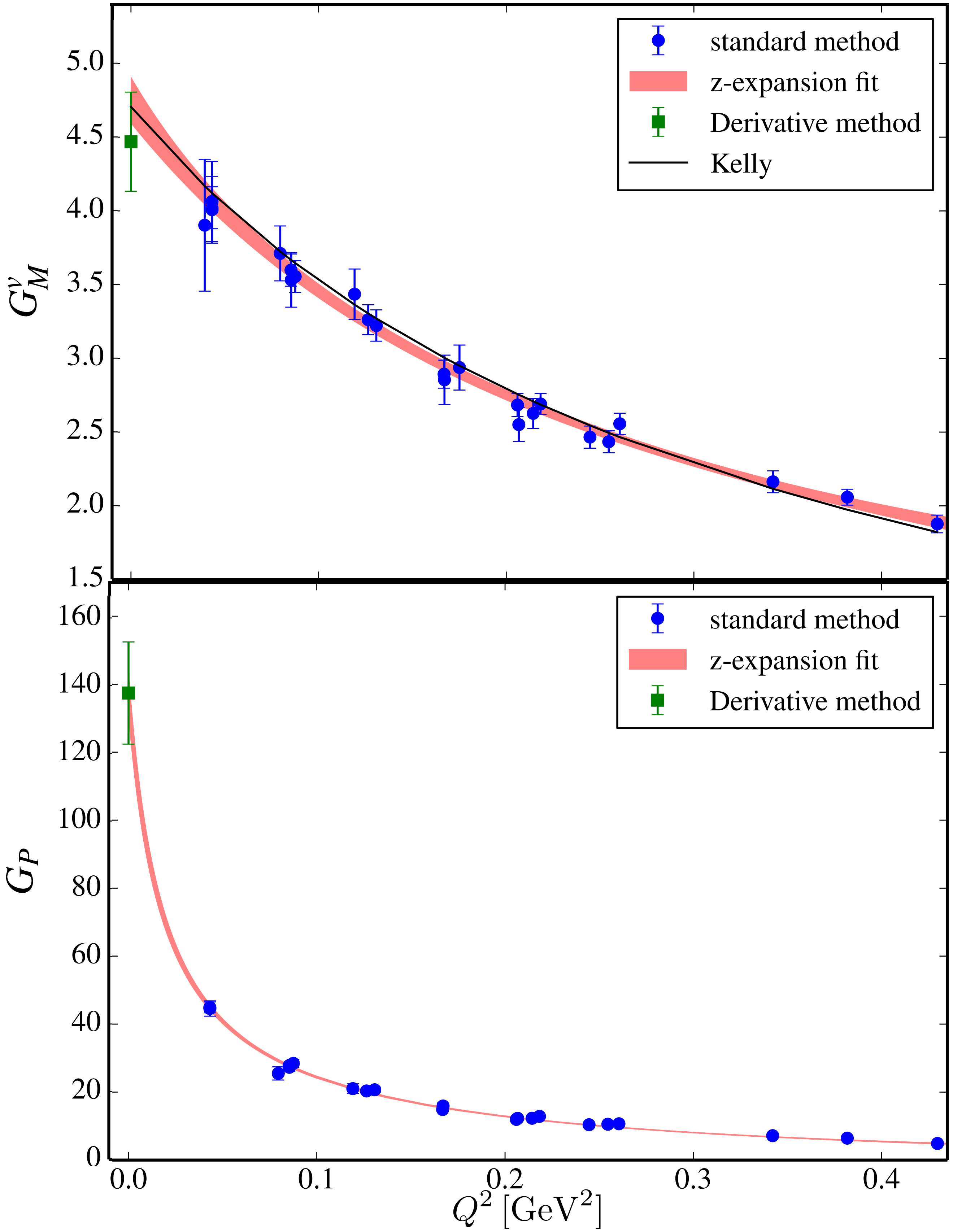
m_π =135 MeV a=0.116 fm 48^4 (BMWc)

[N.Hasan, LATTICE2017]

$$\frac{\frac{\partial}{\partial p'^j}\frac{\partial}{\partial p^j}C_3^0(\vec{p}',\vec{p},T,\tau)|_{\vec{p}=\vec{p}'=0}}{C_2(\vec{0},\tau)} = \frac{1}{4m^2}[F_1 + 2F_2] + \frac{1}{3}F_1[r_1]^2$$



Form Factors Vanishing in Forward Nucleon M.E.



- Sachs magnetic form factor

$$\langle p'|V_\mu|p\rangle = \bar{u}'\left[F_1\gamma_\mu + F_2\frac{\sigma_{\mu\nu}q^\mu}{2m_N}\right]u$$

$$G_M = F_1(Q^2) + F_2(Q^2)$$

$$\partial_1 R_V^2 = -\frac{i}{2m}G_M(0)$$

- Induced pseudoscalar form factors

$$\langle p'|A_\mu|p\rangle = \bar{u}'\left[G_A\gamma_\mu\gamma_5 + G_P\frac{q_\mu\gamma_5}{m_N}\right]u$$

$$G_P(0) = m^2\left(\partial_1^2 R_A^3 + \partial_2^2 R_A^3 - 2\partial_3^2 R_A^3\right)$$

m_π =135 MeV a=0.093 fm 64^4 (BMWc)

[N.Hasan et al, PRD97: 034504 (1711.11385)]

Summary

- Multiple lattice results for nucleon form factors at the physical point
  Chiral extrapolation in m_π no longer required

- Large systematic bias seen by all lattice groups
  Overestimate $G_{Ev}(Q^2=0.4\ \mathrm{GeV}^2)$ by 15-20%
  Underestimate isovector radius by 20-25%

- Precision for charge radius is insufficient for any conclusions
  Both statistical and systematic uncertainty

- Multiple potential sources of systematic uncertainty to explore
  Excited state effects
  Finite volume effects
  Zero-momentum extrapolation

- New promising methods for nucleon structure at zero momentum
  Charge radius
  Form factors vanishing from forward matrix elements