Embedding a Critical Point in a Background Equation of State for QCD

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Purpose and Goals

• Many model calculations predict the existence of a critical point in the QCD phase diagram at a value of the chemical potential where current lattice simulations are unreliable.

• How to combine or merge a critical equation of state with a smooth background is a long-standing problem in statistical physics with no unique solution.

• Our goal is to construct an equation of state in the same universality class as the liquid–gas phase transition and the 3D Ising model. It should have parameters which may be inferred by hydrodynamic modeling of heavy ion collisions in the Beam Energy Scan II at the Relativistic Heavy Ion Collider or in experiments at other accelerators.

• Such an equation of state is also needed for modeling neutron star mergers and closely related to the cold dense matter comprising neutron stars.

• We provide two very different mathematical constructions.

Construction I (JK, TW, CP)

- Motivated by S-shaped curves in first order phase transitions and the cubic equation

$$Q_{\pm}(T,\mu) = \left\{ \left[(\Delta^2(T))^2 + r^2(T,\mu) \right]^{1/2} \pm r(T,\mu) \right\}^k$$

$$r(T,\mu) = \frac{\mu^4 - \mu_x^4(T)}{\mu^4 + \mu_x^4(T)} \qquad \Delta^2(T) \sim d_{\pm} |T/T_c - 1|^p \;\; \text{for} \;\; T \to T_c^{\pm}$$

- Only two exponents k and p

$$P(T,\mu) = P_{BG}(T,\mu) R(T,\mu)$$

- For $T \geq T_c$

$$R(T,\mu) = 1 - a(T) \left(\sqrt{\Delta^4 + 1} + 1 \right)^k - a(T) \left(\sqrt{\Delta^4 + 1} - 1 \right)^k + a(T)(Q_+ + Q_-)$$

- For $T \leq T_c$ and $\mu \leq \mu_x(T)$

$$R_H = 1 + a(T) Q_-(T,\mu) - a(T) \left(\sqrt{\Delta^4 + 1} + 1 \right)^k$$

- For $T \leq T_c$ and $\mu \geq \mu_x(T)$

$$R_Q = 1 + a(T) Q_+(T,\mu) - a(T) \left(\sqrt{\Delta^4 + 1} + 1 \right)^k$$

Critical Behavior I (JK, TW, CP)

- As $n \to n_c$ along the critical isotherm

$$P - P_c \sim \mathrm{sgn}(n - n_c)|n - n_c|^{\delta}\,, \quad \delta = 1/(k-1)$$

- As $t = (T - T_c)/T_c \to 0^+$ the susceptibility and heat capacity are

$$\chi_B \to \chi_+ t^{-\gamma}\,, \quad \gamma = (2-k)p$$
$$c_V \to c_+ t^{-\alpha}\,, \quad \alpha = 2 - kp$$

- As $t \to 0^-$ the susceptibility, heat capacity and density difference along the coexistence curve are

$$\chi_B \to \chi_-(-t)^{-\gamma}$$
$$c_V \to c_-(-t)^{-\alpha}$$
$$\Delta n \sim (-t)^{\beta}\,, \quad \beta = (k-1)p$$

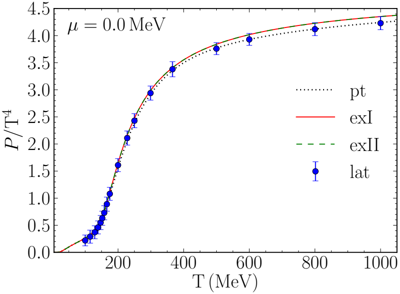
- The critical exponents automatically satisfy the known relations $\alpha + 2\beta + \gamma = 2$ and $\gamma = \beta(\delta - 1)$.

- Predicts relation between universal ratios of critical amplitudes

$$\left(\frac{c_+}{c_-}\right)^{2-k} = 4\left(\frac{\chi_-}{\chi_+}\right)^{k} = 2^{2-k}\left(\frac{d_+}{d_-}\right)^{(2-k)k}$$

Background Equation of State

The background equation of state uses a switching function to transition smoothly from a hadron resonance gas, with excluded volume interactions, to a perturbative quark–gluon plasma. Two parameters in the QCD running coupling, two in the switching function, and an excluded volume parameter are adjusted and fixed by fitting to lattice QCD at $\mu = 0$.



M. Albright, J. Kapusta, and C. Young, Phys. Rev. C (2014)

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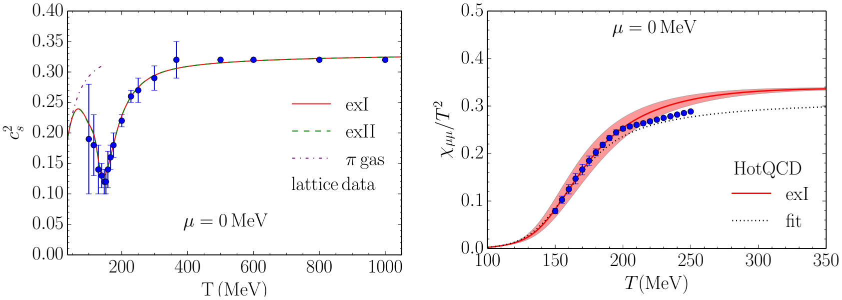

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Illustrative Parameter Choices I (JK, TW, CP)

• In order to have an inverted U-shaped coexistence curve in the $T - n$ plane, as seen in the argon and carbon dioxide liquid-gas phase transitions, the function $\mu_x(T)$ is determined by $R(T, \mu_x(T))n_{BG}(T, \mu_x(T)) = n_c$.

• The critical parameters T_c, μ_c, n_c are related by $R(T_c, \mu_c)n_{BG}(T_c, \mu_c) = n_c$.

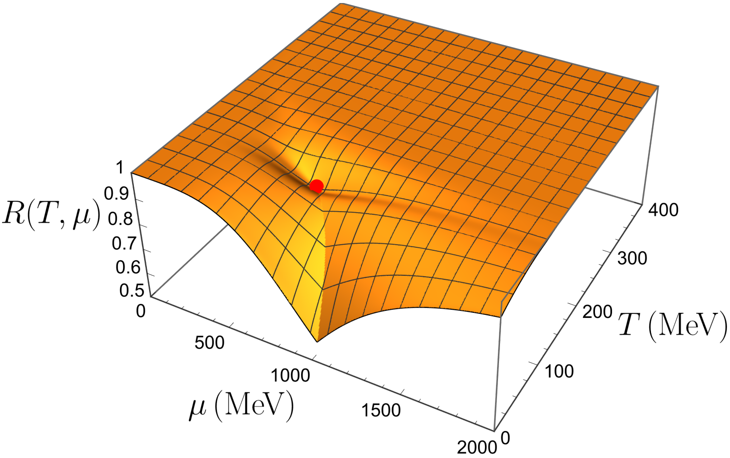
• 3D Ising model exponents give $k = 1.209, p = 1.564$ (mean field values are $k = 4/3, p = 3/2$). Then ratios of critical amplitudes give $d_+/d_- \approx 1/3$ (mean field value is $d_+/d_- = 1$).

$$a(T) = a_0 \exp(-T/T_a)$$

$$\Delta^2(T) = d_+(T/T_c - 1)^p \exp(-T/T_d) \quad T \geq T_c$$

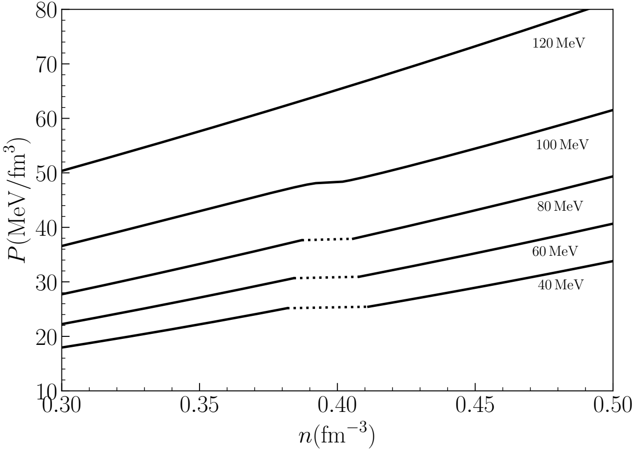
$$\Delta^2(T) = d_-(1 - T/T_c)^p \exp(-T/T_d) \quad T \leq T_c$$

$P(T,\mu) = P_{BG}(T,\mu)R(T,\mu)$ I (JK, TW, CP)



$T_c = 100$ MeV, $\mu_c = 750$ MeV, $n_c \approx 0.4\ \mathrm{fm}^{-3}$

Isotherms of Pressure versus Density I (JK, TW, CP)



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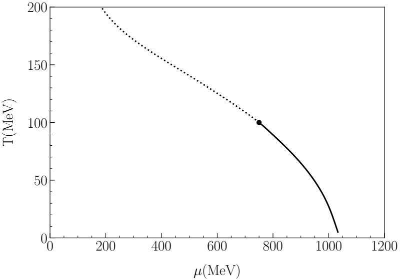
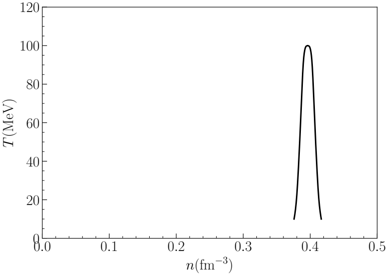
Coexistence Curve I (JK, TW, CP)

$$R(T, \mu_x(T))n_{BG}(T, \mu_x(T)) = n_c$$





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Construction II (JK, TW)

- Adopt the Schofield parametric scaling equation of state (1969)

$$\text{temperature} \quad \frac{T - T_c}{T_c} = R(1-\theta^2)$$

$$\text{magnetization} \quad M \to \frac{n - n_c}{n_c} = m_0 R^\beta \theta$$

$$\text{magnetic field} \quad H \to \frac{\mu - \mu_c}{\mu_c} = h_0 R^{\beta\delta} h(\theta)$$

$$h(\theta) = \theta(1 + h_3\theta^2 + h_5\theta^4)$$

$R \geq 0$ and $-\theta_0 \leq \theta \leq \theta_0$ where $h(\theta_0) = 0$ with $\theta_0 > 1$

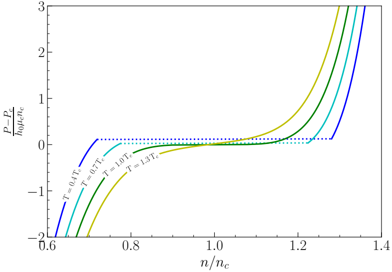
- The pressure must satisfy the condition $(\partial P/\partial\mu)_T = n$ implying

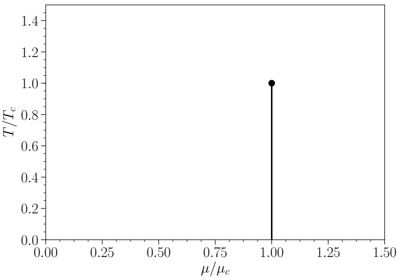
$$P = P_c + [\mu(R,\theta) - \mu_c]n(R,\theta) - m_0 h_0 \mu_c n_c R^{2-\alpha} g(\theta)$$

where $g(\theta)$ is determined by $h(\theta)$

- Critical curve is $\theta = \pm\theta_0$ and critical point is at $R = 0$
- h_3 and h_5 are determined by ratio of critical amplitudes
- h_0 and m_0 are positive parameters

Scaling equation of state only





$$m_0 = 0.2$$

Including the Background II (JK, TW)

• Modify the scaling variables

$$\frac{n - n_c}{n_c} = m_0 R^\beta \theta$$

$$\frac{\mu - \mu_x(T)}{\mu_c} = h_0 R^{\beta\delta} h(\theta)$$

This maintains the density as the order parameter

• Pressure

$$P(\mu, T) = P_{BG}(\mu, T) + W(\mu, T) P_*(R, \theta)$$

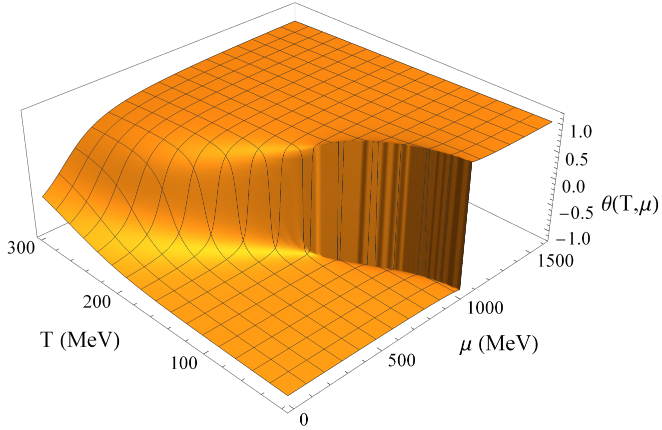
$$P_*(R, \theta) = P_0 + h_0 \mu_c n_0 R^{\beta\delta} h + m_0 h_0 \mu_c n_0 R^{2-\alpha} \left[\theta h(\theta) - g(\theta)\right]$$

• $W(\mu, T)$ is a window function that suppresses the critical contribution away from the coexistence curve

• Background is same as before except for use of point hadrons

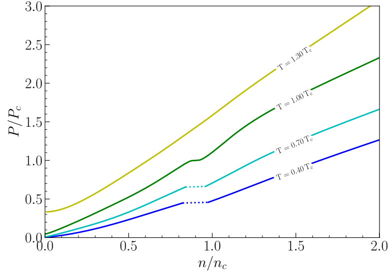
• In order to have an inverted U-shaped coexistence curve in the $T - n$ plane, as seen in the argon and carbon dioxide liquid-gas phase transitions, the function $\mu_x(T)$ is determined by $n_{BG}(\mu_x(T), T) = n_c - n_0$.

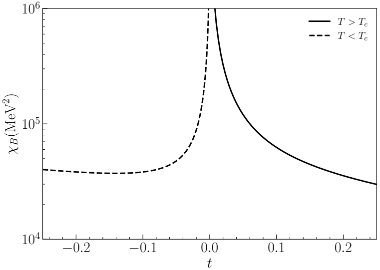
$\theta(T,\mu)$ II (JK, TW)



$T_c = 120$ MeV, $\mu_c = 750$ MeV, $n_c \approx 1.3$ fm^{-3}

Pressure and Susceptibility II (JK, TW)





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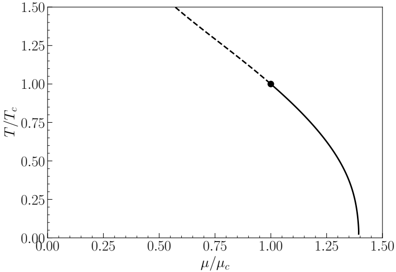
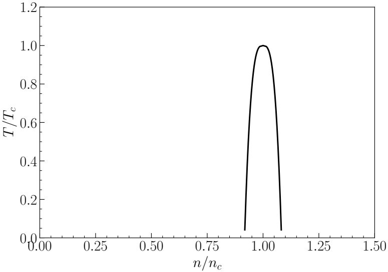
Coexistence Curve II (JK, TW)

$$n_{BG}(\mu_x(T), T) = n_c - n_0$$





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Conclusion

• Lattice QCD simulations have shown unequivocally that the transition from hadrons to quarks and gluons is a crossover when the baryon chemical potential is zero or small. Using two different constructions, we show how to embed a critical point in a smooth background equation of state so as to yield the critical exponents and critical amplitude ratios expected of a transition in the same universality class as the liquid–gas phase transition and the 3D Ising model.

• Apart from the critical exponents and ratios of critical amplitudes (which are universal) and T_c and μ_c, construction I has 4 parameters while construction II has 6.

• The parameters might be inferred by hydrodynamic modeling of heavy ion collisions in the Beam Energy Scan II at the Relativistic Heavy Ion Collider or in experiments at other accelerators.

• With more realistic nuclear interactions, the equations of state may be used to model neutron star mergers.

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Carbon Dioxide and Argon